UNIT I

INTRODUCTION

FUNDAMENTAL CONCEPTS OF ENGINEERING ANALYSIS

OBJECTIVES

The analyst needs certain requirements while designing and assembling the parts of the product. Those requirements are mentioned below

To calculate
i, Displacement at certain points
ii, Stress distribution
iii, Natural frequencies
iv, Critical buckling loads
v, Vibrations
vi, Pressure, velocity and temperature distribution.
vii, Crack growth, residual strength and fatigue life.

METHODS OF ENGINEERING ANALYSIS

There are three different approaches to achieve the above mentioned objectives. They are :

1. Experimental methods
2. Analytical methods
3. Numerical methods or approximate methods.

① EXPERIMENTAL METHODS

In this methods, prototypes can be used. If we want to change the dimensions of the prototype, we have to disassemble the entire prototype and reassemble it and then testing should be carried out. It needs man power and materials. So, it is time consuming and costly process.

② ANALYTICAL METHODS OR THEORETICAL ANALYSIS

In these methods, problems are expressed by mathematical differential equations. It gives quick and closed form solutions. It is used only for simple geometries and idealized support and loading conditions.

③ NUMERICAL METHODS.

Analytical solutions can be obtained only for certain simplified situations. For problems involving complex material properties and boundary conditions, the engineer prefers numerical methods that gives approximate but acceptable solutions. The following three methods are coming under numerical solutions.

(i) Functional Approximation

(ii) Finite Difference Method (FDM)

(iii) Finite Element Method (FEM)

(i) FUNCTIONAL APPROXIMATION:

* The classical methods such as Rayleigh-Ritz methods (variational approach) and Galerkin methods (weighted residual methods) are based on functional approximation but vary in their procedure for evaluating the unknown parameters.

* Rayleigh-Ritz method is useful for solving complex structural problems, encountered in finite element analysis.

* Weighted residual method is useful for solving non-structural problems.

(ii) FINITE DIFFERENTIAL METHOD (FDM)

* Finite difference method is useful for solving heat transfer fluid mechanics and structural mechanics problems. It is a general method. It is applicable to any phenomenon for which differential equation along with the boundary conditions are available. It works well for two dimensional regions. with boundaries parallel to the coordinate axes.

* The starting point in the finite difference method is that the differential equation must be known before solving. After that, the region is subdivided into a convenient number of divisions. The differential equation is applied successively at the various points of the subdivided

region, a set of simultaneous equations are generated which upon solving lead to approximate solution to the problem. This is the essence of finite difference method.

* This method is difficult to use when regions have curved or irregular boundaries and it is difficult to write general computer programs.

(iii) FINITE ELEMENT METHOD (FEM) OR FINITE ELEMENT ANALYSIS (FEA)

* Finite element method is a numerical method for solving problems of Engineering and Mathematical Physics.

* In this method, a body or a structure in which the analysis to be carried out is subdivided into smaller elements of finite dimensions called finite elements. Then the body is considered as an assemblage of these elements connected at a finite number of joints called 'Nodes' or Nodal points. The properties of each type of finite element is obtained and assembled together and solved as whole to get solution.

* In other words, in the finite element method, instead of solving the problem for the entire body in one operation, we formulate the equations for each finite element and combine them to obtain the solution of the whole body.

* Finite element method is used to solve physical problems involving complicated geometrics, loading and material properties which cannot be solved by analytical method. This method is extensively used in the field of structural mechanics, fluid mechanics, heat transfer, mass transfer, electric and magnetic field problems.

Based on application, the finite element problems are classified as follows:

(i) Structural problems.

(ii) Non-structural problems.

(i) STRUCTURAL PROBLEMS:

In structural problems, displacement at each nodal point is obtained. By using these displacement solutions, stress and strain in each element can be calculated.

(ii) NON-STRUCTURAL PROBLEMS:

In non-structural problems, temperature or fluid pressure at each nodal point is obtained. By using these values, properties such as heat flow, fluid flow etc., for each element can be calculated.

HISTORICAL BACKGROUND OF FEM

* Basic ideas of the finite element analysis were developed by aircraft engineers in the early 1940s. These were primarily the matrix methods of analysis.

* The modern development of the finite element method began in the year of 1945 in the field of structural engineering with the work by Hrennikoff.

* In 1947 Levy introduced the flexibility or force method and in 1953 he suggested stiffness method which could be a promising alternative for use in analysing statically redundant aircraft structures.

* By using energy principles, Argyris and kelsey developed matrix structural analysis methods in 1954. This development illustrated the important role that energy principles would play in the finite element method.

* The term finite element was first introduced by Clough in 1960 in the plane stress analysis and he used both triangular and rectangular elements in that analysis.

* Most of the finite element work upto early 1960s dealt with small strains and small displacements, elastic material behaviour and static loadings. In 1961, Turner considered large deflection and thermal analysis problems. In 1962, Gallagher introduced material non-Linearities problems, whereas

buckling problems were initially treated by Gallagher and Padlog in 1963. In 1968, Zinkiewicz extended the method to visco elasticity problems.

* Weighted residual methods was first introduced by Szabo and Lee in 1969 for structural analysis and then by Zinkiewicz and Parekh in 1970 for transient field problems.

* During the decades of the 1960s and 1970s, the finite element method was extended to applications in shell bending, plate bending, heat transfer analysis, fluid flow analysis and general three dimensional problems in structural analysis.

* From the early 1950s to present, enormous advances have been made in the application of finite element method to solve complicated engineering problems. It is curious to note that the present day finite element method does not have its root in one discipline. The mathematicians contime to put the finite element method on sound theoretical ground whereas the engineers contime to find interesting extensions in various branches of engineering. These concurrent developments have made the finite element method as one of the most powerful approximate methods.

GENERAL STEPS OF THE FINITE ELEMENT ANALYSIS:

* This section presents the general procedure of finite element analysis. For simplicity's sake, we will consider only the structural problems.

* The following two general methods are associated with the finite element analysis. They are

(i) Force method

(ii) Displacement or stiffness method

* In force method, internal forces are considered as the unknowns of the problem. In displacement or stiffness methods, displacements of the nodes are considered as the unknowns of the problem.

* Among these two approaches, displacement method is more desirable because its formulation is simpler for most structural analysis problems. So, a vast majority of general purpose finite element programs have used the displacement formulation for solving structural problems.

* We now present the steps along with explanations used in the finite element method formulation.

STEP 1: Discretization of Structure

The art of subdividing a structure into a convenient number of smaller elements is known as discretization.

Smaller elements are classified as follows:

(i) One dimensional elements
(ii) Two dimensional elements
(iii) Three dimensional elements
(iv) Axisymmetric elements

STEP 2: Numbering of Nodes and Elements

The nodes and elements should be numbered after discretization process. The numbering process is most important since it decide the size of the stiffness matrix. and it leads the reduction of memory requirement. While numbering the nodes, the following condition should be satisfied.

Maximum node number − Minimum node number = Minimum

STEP 3: Selection of a Displacement Function or Interpolation Function

It involves choosing a displacement function within each element. Polynomial of linear, quadratic and cubic form are frequently used as displacement functions because they are simple to work within finite element formulation.

The polynomial type of interpolation functions are mostly used due to the following reasons.

1. It is easy to formulate and computerize the finite element equations.
2. It is easy to perform differentiation or integration.
3. The accuracy of the results can be improved by increasing the order of the polynomial.

Let us consider $\phi(x)$ is a field variable

Case (i): Linear Polynomial:

One dimensional problem $\phi(x) = a_0 + a_1 x$

Two dimensional problem $\phi(x,y) = a_0 + a_1 x + a_2 y$

Three dimensional problem $\phi(x,y,z) = a_0 + a_1 x + a_2 y + a_3 z$

Case (ii): Quadratic Polynomial:

One dimensional problem $\phi(x) = a_0 + a_1 x + a_2 x^2$

Two dimensional problem $\phi(x,y) = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 y^2 + a_5 xy$

Three dimensional problem $\phi(x,y,z) = a_0 + a_1 x + a_2 y + a_3 z + a_4 x^2 + a_5 y^2 + a_6 z^2 + a_7 xy + a_8 yz + a_9 xz$

STEP 4: Define the material behaviour by using Strain-Displacement and Stress-Strain Relationships

* Strain-Displacement and Stress-Strain relationships are necessary for deriving the equations for each finite element.

* In case of one dimensional deformation, the strain-displacement relationship is given by,

$$e = \frac{du}{dx}$$

where, $u \rightarrow$ Displacement field variable along x direction

$e \rightarrow$ Strain

The stress-strain relationship is given by,

$$\sigma = Ee$$

where, $\sigma \rightarrow$ Stress in x direction

$E \rightarrow$ Modulus of elasticity or Young's modulus

STEP 5: Derivation of element stiffness matrix and equations.

The finite element equation is in matrix form as,

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ \vdots \\ F_n \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & \cdots & k_{1n} \\ k_{21} & k_{22} & k_{23} & \cdots & k_{2n} \\ k_{31} & k_{32} & k_{33} & \cdots & k_{3n} \\ \vdots & & & & \vdots \\ k_{n1} & \cdots & \cdots & \cdots & k_{nn} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{Bmatrix}$$

In compact matrix form as,

$$\{F^e\} = [k^e]\{u^e\}$$

where, e is a Element, $\{F\}$ is the vector of element nodal forces, $[k]$ is the element stiffness matrix and $\{u\}$ is the element displacement vector.

This equation can be derived by any one of the following methods

(i) Direct Equilibrium Method: This method is much easier to apply for line or one dimensional elements.

(ii) Variational Method: This method is most easily adaptable to the determination of element equations for complicated elements (i.e, element having large number of degrees of freedom) like axisymmetric stress element, plate bending element and two or three dimensional solid stress element.

(iii) Weighted Residual Method: This method is (Galerkin's method) useful for developing the element equations in thermal analysis problems. They are especially useful when a functional such as potential energy is not readily available.

STEP 6: Assemble the element equations to obtain the global or total equations.

The individual element equations obtained in step 5 are added together by using a method of superposition i.e., direction stiffness method. The final assembled or global equation which is in the form of

$$\{F\} = [K]\{u\}$$

where, $\{F\} \rightarrow$ Global force vector

$[K] \rightarrow$ Global stiffness matrix

$\{u\} \rightarrow$ Global displacement vector.

STEP 7: Applying boundary conditions.

From equation $\{F\} = [K]\{u\}$ we know that global stiffness matrix $[K]$ is a singular matrix because its determinant is equal to zero. In order to remove this singularity problem, certain boundary conditions are applied so that the structure remains in place instead of moving as a rigid body. The global equation to be modified to account for the boundary conditions of the problem.

STEP 8: Solution for the unknown displacements

A set of simultaneous algebraic equations formed in step 6 can be written in expanded matrix form as follows:

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ \vdots \\ \vdots \\ F_n \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & \cdots & k_{1n} \\ k_{21} & k_{22} & k_{23} & \cdots & k_{2n} \\ k_{31} & k_{32} & k_{33} & \cdots & k_{3n} \\ k_{41} & k_{42} & k_{43} & \cdots & k_{4n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ k_{n1} & k_{n2} & k_{n3} & \cdots & k_{nn} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ \vdots \\ \vdots \\ u_n \end{Bmatrix}$$

These equation can be solved and unknown displacements $\{u\}$ are calculated by using Gaussian elimination method or Gauss - Seidel method

STEP 9: Computation of the element strains and stresses from the nodal displacements, $\{u\}$:

In structural stress analysis problem, stress and strain are important factors. From the solution of displacement vector $\{u\}$, stress and strain value can be calculated.

In case of one dimensional deformation, the strain-displacement relationship is given by

$$\text{Strain, } e = \frac{du}{dx}$$

$$= \frac{u_2 - u_1}{x_2 - x_1}$$

where u_1 and u_2 are displacement at node 1 and 2.

$x_2 - x_1$ = Actual length of the element

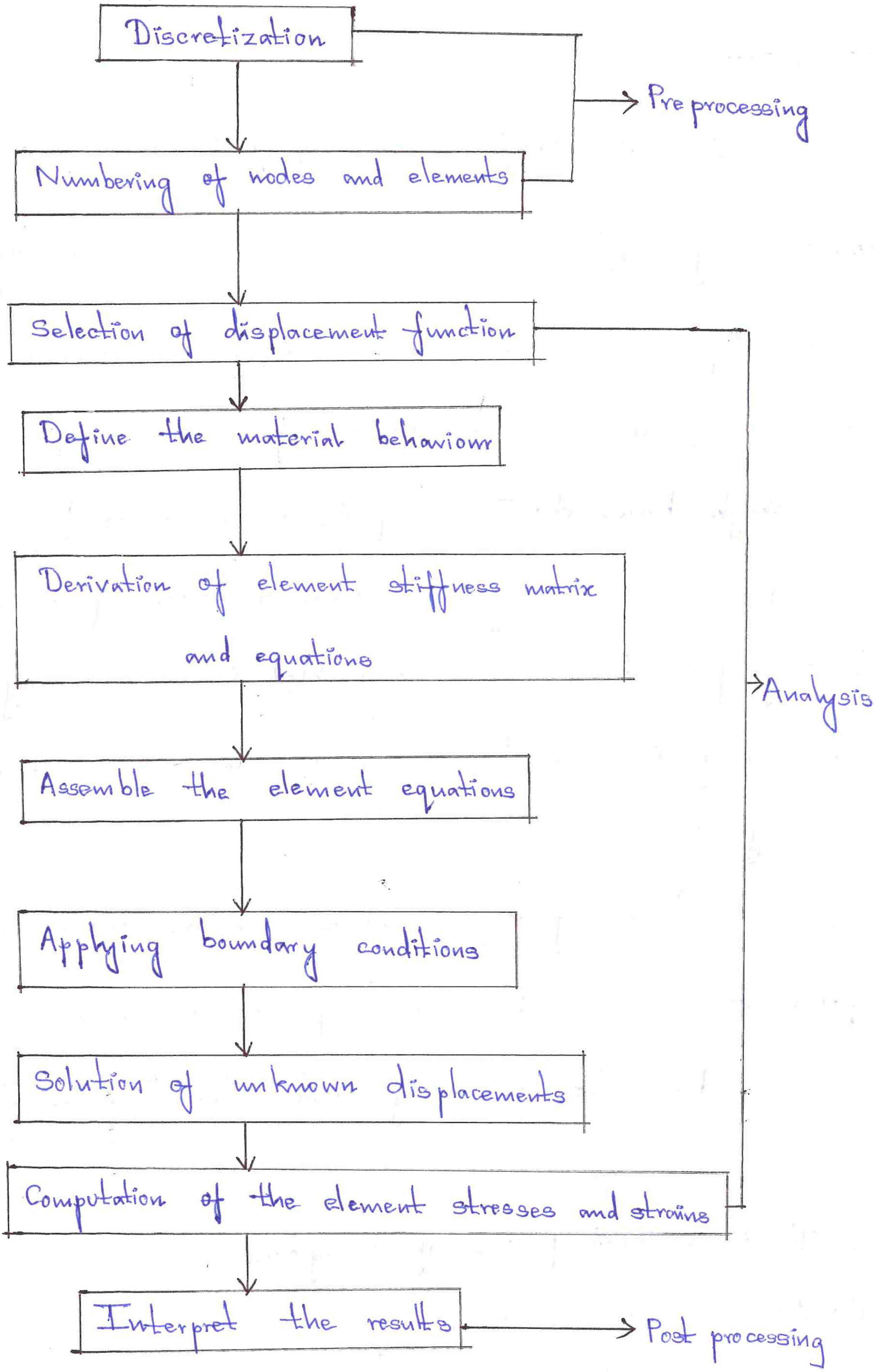
From that, we can find the strain value.

By knowing the strain, stress value can be calculated by using the relation, Stress $\sigma = Ee$ where $E \rightarrow$ Young's modulus, $e \rightarrow$ Strain.

STEP 10: Interpret the results (Post Processing):

Analysis and evaluation of the solution results is referred to as post-processing. Post processor computer programs help the user to interpret the results by displaying them in graphical form.

STEP 1 to 10 are summarized as follows:


Discretization
Numbering of nodes and elements
Pre processing
Selection of displacement function
Define the material behaviour
Derivation of element stiffness matrix and equations
Analysis
Assemble the element equations
Applying boundary conditions
Solution of unknown displacements
Computation of the element stresses and strains
Interpret the results
Post processing

DISCRETIZATION:

INTRODUCTION:

In this chapter we are going to learn about discretization, node, assembly, system etc. To make this much easier to understand, Let us compare these words with the parts over human body. Apart from flesh, our body consists of bones. They are hands, legs, fingers, thigh bones, etc. These parts are connected together at different places, so that when movement takes place, we do not feel any pain. Nature has assembled in such a way that every human being is able to sustain certain amount of load without experiencing stain.

Similarly any structure like an automobile, ship, aeroplane, etc., consists of several components assembled together.

Now let us study about 'Element'. The characteristics of an element are as follows:

(i) It is a small portion of a system

(ii) It has definite shape

(iii) It should have minimum two nodes.

(iv) Nodes are placed where connection is made to another element.

(v) Loads act only at the nodes.

DISCRETIZATION:

The art of subdividing a structure into a convenient number of smaller components is known as discretization. These smaller components are then put together. The process of uniting the various elements together is called Assemblage. The assemblage of such elements then represents the original body. Discretization can be classified as follows:

(i) Natural

(ii) Artificial (continuum)

NATURAL DISCRETIZATION

In structural analysis a truss is considered as a natural system. The various members of the truss constitute the elements. These elements are connected at various joints known as nodes.

NODAL POINTS:

Each kind of finite element has a specific structural shape and is interconnected with the adjacent elements by nodal points or nodes.

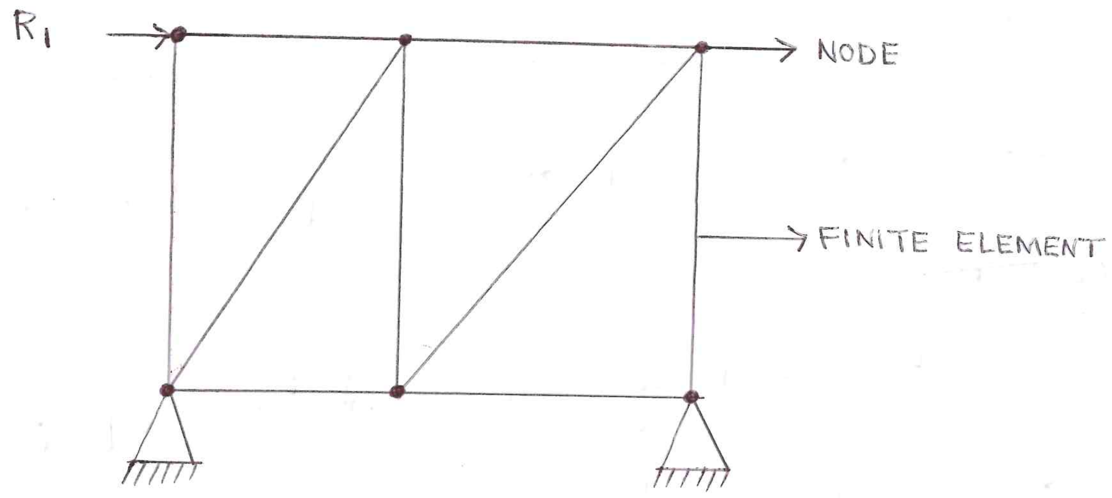
NODAL FORCES:

The forces that act at each nodal point are called nodal forces.

DEGREES OF FREEDOM:

When the force or reaction act at nodal point, node is subjected to deformation This deformation includes displacements, rotations, and /or strains. These are collectively known as degrees of freedom or simply we can say nodal displacement is called degrees of freedom.

In figure the truss consists of 9 elements and 6 nodes. There are four freely moving and two extreme constrained nodes. The truss is a natural system as there is no possibility either to increase or decrease the number of elements and the nodes.



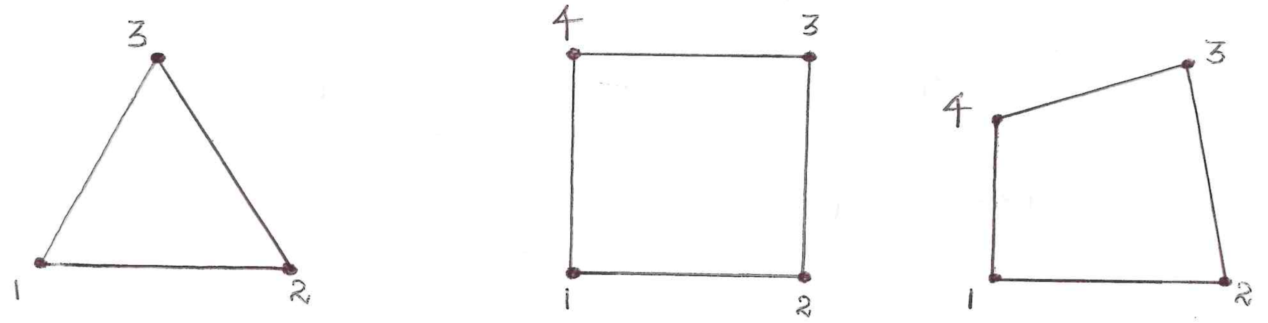
NATURAL DISCRETIZATION OF TRUSS

ARTIFICAL DISCRETIZATION (CONTINUUM)

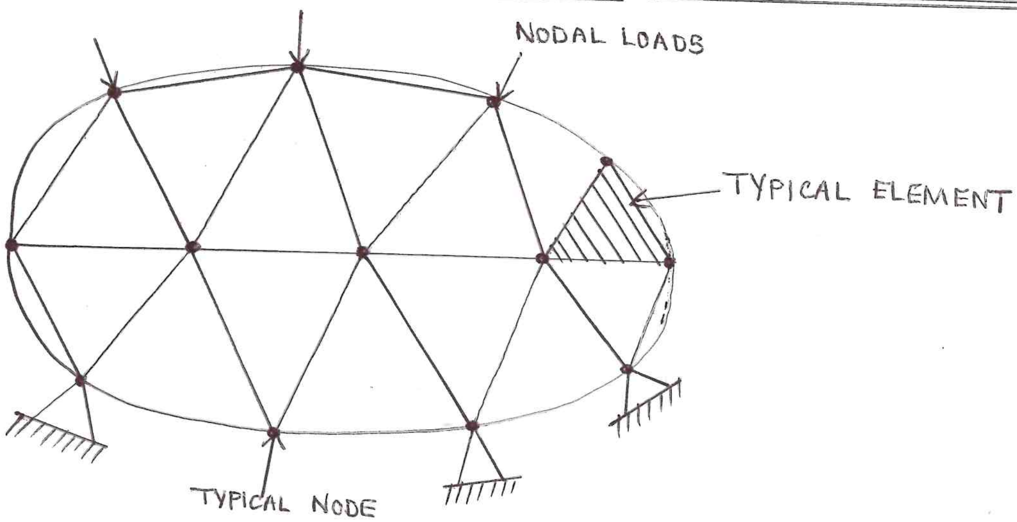
Continuum is generally considered to be a single mass of material as found in a forging, concrete dam, deep beam, plate and so on.

Unlike the truss element which is physically present in the truss, in a continuum, the following three elements exist only in our imagination.

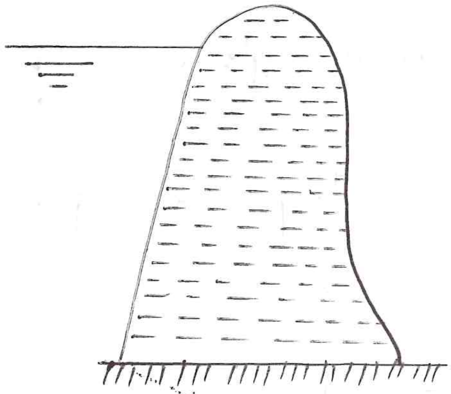
1. Triangular element
2. Rectangular element
3. Quadrilateral element



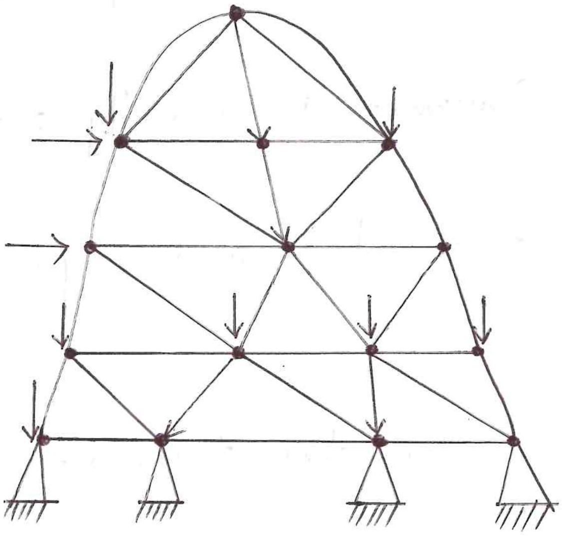
TRIANGULAR ELEMENT | RECTANGULAR ELEMENT | QUADRILATERAL ELEMENT



DISCRETIZATION USING TRIANGULAR ELEMENTS



GRAVITY DAM



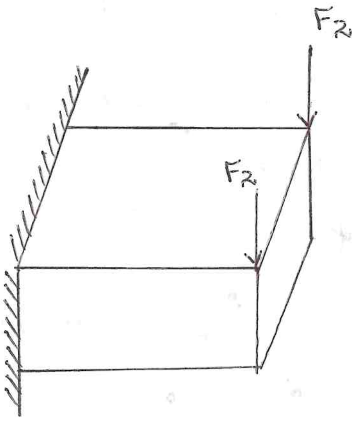
DISCRETIZATION USING TRIANGULAR ELEMENTS

DISCRETIZATION PROCESS:

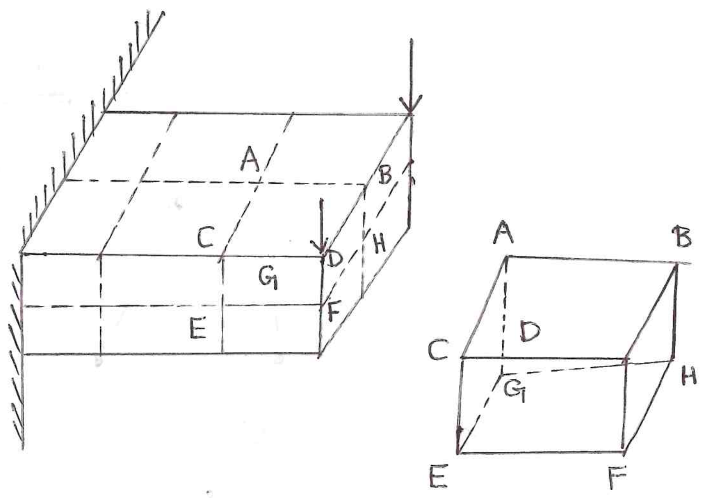
The following points to be considered while analysing the discretization process.

(i) TYPE OF ELEMENTS:

The type of elements to be used will be evident from the physical problem.



SHORT BEAM



DISCRETIZATION USING 3D ELEMENTS

The choice of the element to be used for discretization depends upon the following factors.

(i) Number of degrees of freedom needed

(ii) Expected accuracy

(iii) Necessary equations required.

However in certain problems the given structure cannot be discretized by using only one type of elements. In such cases, we can use two or more types of elements for discretization. Example: Air craft wing.

(ii) SIZE OF ELEMENTS

* The size of elements influences the convergence of the solution of the problems directly. So, it should be chosen with more care.

* If the size of the element is small, the final solution is more accurate. But the computational time for the smaller size element is more when compared to larger size element.

* Another characteristic related to the size of elements that affects the finite element problem solution is the Aspect ratio of the elements.

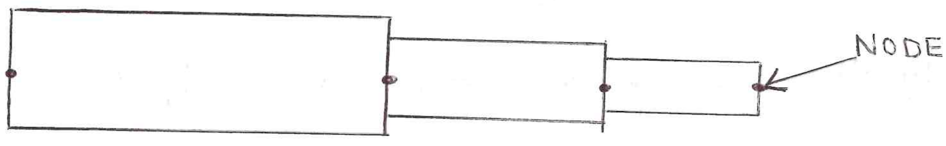
* Aspect ratio is defined as the ratio of the largest dimension of the element to the smallest dimension. The conclusion of many researchers is that the aspect ratio should be close to unity as

possible. For a two dimensional rectangular element, the aspect ratio is conveniently defined as length to breadth ratio. Aspect ratio closer to unity yields better results.

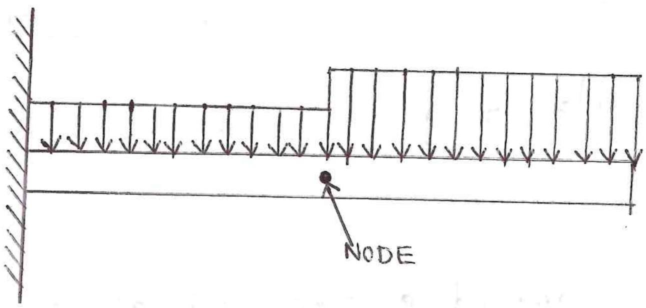
(iii) LOCATION OF NODES:

* If the structure has no abrupt changes in geometric, load, boundary conditions and material properties, the structure can be divided into equal subdivisions. So, the spacing of the nodes are uniform.

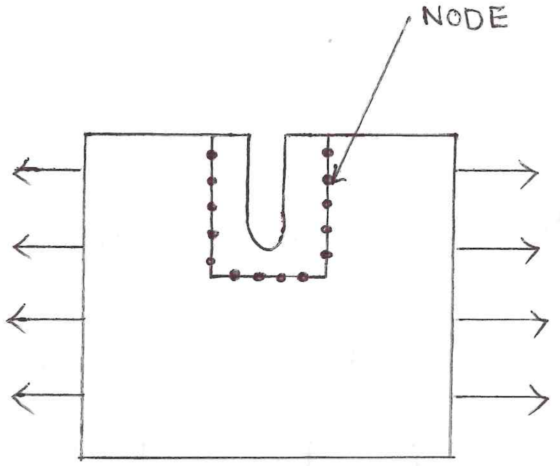
* If there are any discontinuities in geometric, load, boundary conditions and material properties of the structure, nodes should be introduced at these discontinuities as shown in the following figure.



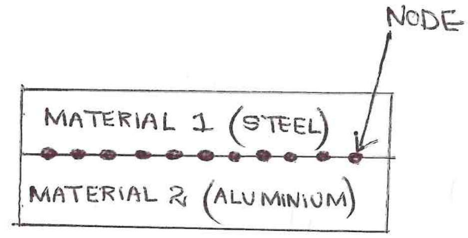
GEOMETRIC DISCONTINUITIES



DISCONTINUITY IN LOADING



CRACKED PLATE UNDER LOADING



MATERIAL DISCONTINUITY

DISCONTINUITY OF BOUNDARY CONDITIONS

(iv) NUMBER OF ELEMENTS:

The number of elements to be selected for discretization depends upon the following factors:

1. Accuracy desired
2. Size of the elements
3. Number of degrees of freedom involved

If the number of element in the structure is increased, the final solution of the problem is expected to be more accrate. But the use of large number of elements involves a large number of degrees of freedom, it leads the storage problem in the available computer memory.

GENERAL PROCEDURE:

Our interest is to find y, which is the solution for the differential equation. If it is not possible to find a solution, we assume an approximate function for y. When we substitute the approximate solution in the differential equation, we can get residual and that residual can be expressed as

$$R\left(x_i\,;\, a_1, a_2, a_3\right) = 0$$

where a_1, a_2 are unknown parameters present in assumed trial function.

The assumed trial function can be expressed as follows:

$$y = f\left(x\,;\, a_1, a_2, a_3, \ldots, a_n\right)$$

Trial function y must exactly satisfy the boundary conditions.

The method of weighted residuals needs the parameters $a_1, a_2, a_3, \ldots, a_n$ to be determined by satisfying the following equation.

$$\int_D W_i\, R\left(x\,;\, a_1, a_2, a_3, \ldots a_n\right) dx = 0 \rightarrow (1)$$

where W_i is a function of x and known as weighting function.

D is a domain ; R is a residual.

POINT COLLOCATION METHOD:

In the collocation method, also called point collocation, residuals are set to zero at n different locations X_i and the weighting function w_i is denoted as $\delta(x-x_i)$.

$$\Rightarrow w_i = \delta(x-x_i)$$

Substituting w_i value in equation (1)

$$\Rightarrow \int_D \delta(x-x_i)\, R(x; a_1, a_2, a_3, \ldots, a_n)\, dx = 0 \rightarrow (2)$$

The x_i's are referred to as collocation points and are selected by the discretion of the analyst.

In equation (2) term $\int_D \delta(x-x_i) = 1$

So, $R(x; a_1, a_2, a_3, \ldots, a_n) = 0$.

SUBDOMAIN COLLOCATION METHOD:

In this method the weighting functions (w_i) are chosen to be unity over a portion of the domain and zero elsewhere. It is given as follows:

$$w_1 = \begin{cases} 1 & \text{for } x \text{ in } D_1 \\ 0 & \text{for } x \text{ not in } D_1 \end{cases}$$

$$w_2 = \begin{cases} 1 & \text{for } x \text{ in } D_2 \\ 0 & \text{for } x \text{ not in } D_2 \end{cases}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$w_n = \begin{cases} 1 & \text{for } x \text{ in } D_n \\ 0 & \text{for } x \text{ not in } D_n \end{cases}$$

where D is a domain

LEAST SQUARES METHOD:

In this method, the integral of the weighted square of the residual over the domain is required to be minimum.

i.e., $I = \int_D \left[R\,(x;\, a_1, a_2, a_3, \ldots, a_n)\right]^2 dx$ = minimum.

where $I = f(a_1, a_2, \cdots a_n)$

The requirement is $\dfrac{\partial I}{\partial a_i} = 0, \quad i = 1, 2, 3, \ldots, n$

GALERKIN'S METHOD

In this method the trial function $N_i(x)$ itself is considered as the weighting functions; that is

$$w_i = N_i(x)$$

Substitute w_i value in equation (1)

$$\Rightarrow \int_D N_i(x)\, R\,(x;\, a_1, a_2, \ldots, a_n)\, dx = 0 \quad \rightarrow (3)$$

$$i = 1, 2, 3, \ldots, n$$

PROBLEMS

① The following differential equation is available for a physical phenomenon $AE\frac{d^2y}{dx^2} + q_0 = 0$ with the boundary conditions

$$y(0) = 0$$

$$\left.\frac{dy}{dx}\right|_{x=L} = 0$$

Find the value of $f(x)$ using the weighted residual method.

GIVEN DATA:

Differential equation $AE\frac{d^2y}{dx^2} + q_0 = 0$

Boundary conditions are $y(0) = 0$

$$\left.\frac{dy}{dx}\right|_{x=L} = 0$$

TO FIND:

$f(x)$

SOLUTION:

Assume a trial solution,

$$\text{Let } y(x) = a_0 + a_1 x + a_2 x^2 \longrightarrow ①$$

Apply first boundary condition

$x = 0$, $y = 0$ in eqn ①

$$0 = a_0 + 0 + 0$$

$$a_0 = 0$$

Apply second boundary condition

$$y(x) = a_0 + a_1 x + a_2 x^2$$

$$\frac{dy}{dx} = a_1 + 2a_2 x$$

At $x = L$, $\frac{dy}{dx} = 0$

$$0 = a_1 + 2a_2 L$$

$$a_1 = -2a_2 L$$

Substituting a_0 and a_1 value in equation (1)

$$y(x) = -2a_2 xL + a_2 x^2$$

$$y(x) = a_2\left[x^2 - 2xL\right] \rightarrow (2)$$

$$\frac{dy}{dx} = a_2\left[2x - 2L\right]$$

$$\frac{d^2y}{dx^2} = 2a_2$$

Residual $R = AE\frac{d^2y}{dx^2} + q_0 = 0$

$$AE(2a_2) + q_0 = 0$$

$$AE\,2a_2 = -q_0$$

$$a_2 = \frac{-q_0}{2AE}$$

Substituting value of a_2 in equation (2)

$$y(x) = \frac{-q_0}{2AE}\left[x^2 - 2xL\right]$$

$$y(x) = \frac{q_0}{2AE}\left[2xL - x^2\right] \quad \boxed{\text{Final solution}}$$

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(2) The governing differential equation for the fully developed laminar flow is given by $\mu \frac{d^2u}{dx^2} + \rho g \cos\theta = 0$. If boundary conditions are $\left.\frac{du}{dx}\right|_{x=0} = 0$, $u(L) = 0$, find the velocity distribution, $u(x)$.

GIVEN DATA:

DE $\mu \frac{d^2u}{dx^2} + \rho g \cos\theta = 0$

Boundary conditions $\left.\frac{du}{dx}\right|_{x=0} = 0$

$u(L) = 0$

TOFIND:

$u(x)$

SOLUTION:

Assume a trial function

Let $u(x) = a_0 + a_1 x + a_2 x^2 \ldots\ldots \rightarrow (1)$

Apply first boundary condition

$$\frac{du}{dx} = 0 \quad \text{at} \quad x = 0$$

$$\frac{du}{dx} = a_1 + 2a_2 x$$

At $x = 0$, $\frac{du}{dx} = 0$

$$a_1 = 0$$

Apply secondary boundary condition

at $x = L$, $u(x) = 0$

$$u(x) = a_0 + a_1 x + a_2 x^2$$

$$0 = a_0 + a_1 L + a_2 L^2$$

Substitute $a_1 = 0$

$$0 = a_0 + a_2 L^2$$

$$a_0 = -a_2 L^2$$

Substituting a_0 and a_1 value in equation (1)

$$u(x) = -a_2 L^2 + 0 + a_2 x^2$$

$$u(x) = a_2 \left[x^2 - L^2\right] \longrightarrow (2)$$

$$\frac{du}{dx} = 2a_2 x$$

$$\frac{d^2u}{dx^2} = 2a_2$$

$$\text{Residual } R = \mu \frac{d^2u}{dx^2} + \rho g \cos\theta = 0$$

$$\mu(2a_2) + \rho g \cos\theta = 0$$

$$\mu\, 2a_2 = -\rho g \cos\theta$$

$$a_2 = \frac{-\rho g \cos\theta}{2\mu}$$

Substitute a_2 value in equation (2)

$$u(x) = \frac{-\rho g \cos\theta}{2\mu}\left[x^2 - L^2\right]$$

$$u(x) = \frac{\rho g \cos\theta}{2\mu}\left[L^2 - x^2\right] \quad \text{[Final solution]}$$

---

(3) Find the solution for the following differential equation.

$$EI \frac{d^4u}{dx^4} - q_0 = 0$$

The boundary conditions are $u(0) = 0$, $\frac{du}{dx}(0) = 0$, $\frac{d^2u}{dx^2}(L) = 0$, $\frac{d^3u}{dx^3}(L) = 0$

GIVEN DATA:

$$\text{DE} \quad EI \frac{d^4u}{dx^4} - q_0 = 0$$

Boundary conditions are $u(0) = 0$, $\frac{du}{dx}(0) = 0$

$\frac{d^2u}{dx^2}(L) = 0$, $\frac{d^3u}{dx^3}(L) = 0$

SOLUTION:

Assume a trial function

Let $u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \ldots \rightarrow (1)$

Apply first boundary condition

at $x = 0$, $u(x) = 0$

$$0 = a_0 + 0 + 0 + 0$$

$$a_0 = 0$$

Apply second boundary condition

at $x = 0$, $\frac{du}{dx} = 0$

$$\frac{du}{dx} = 0 + a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3$$

$$0 = a_1 + 0 + 0 + 0$$

$$a_1 = 0$$

Apply third boundary condition

at $x = L$, $\frac{d^2u}{dx^2} = 0$

$$\frac{d^2u}{dx^2} = 2a_2 + 6a_3 x + 12a_4 x^2$$

$$0 = 2a_2 + 6a_3 L + 12a_4 L^2$$

$$2a_2 = -6a_3 L - 12a_4 L^2$$

$$a_2 = -\left[3a_3 L + 6a_4 L^2\right]$$

Apply fourth boundary condition

at $x = L$, $\frac{d^3u}{dx^3} = 0$

$$\frac{d^3u}{dx^3} = 0 + 6a_3 + 24a_4 x$$

$$0 = 6a_3 + 24a_4 L$$

$$6a_3 = -24a_4 L$$

$$a_3 = -4a_4 L$$

Substitute a_0, a_1, a_2 and a_3 values in equation (1)

$$u(x) = 0 + 0 - \left[3a_3 L + 6a_4 L^2\right]x^2 - 4a_4 Lx^3 + a_4 x^4$$

$$= -\left[3a_3 L + 6a_4 L^2\right]x^2 - 4a_4 Lx^3 + a_4 x^4$$

$$= -\left[3(-4a_4 L)\times L + 6a_4 L^2\right]x^2 - 4a_4 Lx^3 + a_4 x^4$$

$$\left[\because a_3 = -4a_4 L\right]$$

$$= 12a_4 L^2 x^2 - 6a_4 L^2 x^2 - 4a_4 Lx^3 + a_4 x^4$$

$$= a_4\left[12L^2x^2 - 6L^2x^2 - 4Lx^3 + x^4\right]$$

$$u(x) = a_4\left[6L^2x^2 - 4Lx^3 + x^4\right]$$

$$\frac{du}{dx} = a_4\left[6L^2(2x) - 4L(3x^2) + 4x^3\right]$$

$$\frac{d^2u}{dx^2} = a_4\left[6L^2(2) - 4L(6x) + 12x^2\right]$$

$$\frac{d^3u}{dx^3} = a_4\left[0 - 24L + 24x\right]$$

$$\frac{d^4u}{dx^4} = a_4\left[0 - 0 + 24\right]$$

$$\frac{d^4u}{dx^4} = 24a_4$$

Residual $R = EI\dfrac{d^4u}{dx^4} - q_0 = 0$

$$EI(24a_4) - q_0 = 0$$

$$EI\,24a_4 = q_0$$

$$a_4 = \frac{q_0}{24EI}$$

Substitute a_4 value in equation (2)

$$u(x) = \frac{q_0}{24EI}\left[6L^2x^2 - 4Lx^3 + x^4\right]$$

$$u(x) = \frac{q_0}{24EI}\left[x^4 - 4Lx^3 + 6L^2x^2\right]$$ [Final solution]

(4) The following differential equation is available for a physical phenomenon

$$\frac{d^2y}{dx^2} + 50 = 0 \; , \; 0 \leq x \leq 10$$

Trial function is, $y = a_1 x (10 - x)$

Boundary conditions are $y(0) = 0$; $y(10) = 0$

Find the value of the parameter a_1 by the following methods:

(i) Point collocation (ii) Subdomain collocation (iii) Least squares
(iv) Galerkin

GIVEN DATA:

DE $$\frac{d^2y}{dx^2} + 50 = 0 \; ; \; 0 \leq x \leq 10 \quad \rightarrow (1)$$

Trial function $y = a_1 x (10 - x)$

Boundary conditions are $y(0) = 0$
$y(10) = 0$

TOFIND:

Value a_1 by

(i) Point collocation method

(ii) Subdomain collocation method

(iii) Least squares method

(iv) Galerkin method

SOLUTION:

First verify whether the trial function satisfies the boundary conditions or not

Trial function is $y = a_1 x(10-x)$

when $x = 0$, $y = 0$

$x = 10$, $y = 0$

It satisfies the boundary conditions

(i) POINT COLLOCATION METHOD:

$$y = a_1 x(10-x)$$

$$y = 10a_1 x - a_1 x^2$$

$$\frac{dy}{dx} = 10a_1 - 2a_1 x$$

$$\frac{d^2y}{dx^2} = -2a_1$$

Substitute $\frac{d^2y}{dx^2}$ in equation (1)

Residual $R = -2a_1 + 50 \rightarrow (2)$

In point collocation method $R = 0$

$$R = -2a_1 + 50 = 0$$

$$-2a_1 = -50$$

$$a_1 = 25 \rightarrow (3)$$

Trial function is $y = 25x(10-x)$

(ii) SUBDOMAIN COLLOCATION METHOD

This method requires $\int_0^{10} R \, dx = 0$

Substitute R value $\int_0^{10} [-2a_1 + 50] \, dx = 0$

$$\int_0^{10} [-2a_1 \, dx + 50 \, dx] = 0$$

$$\left[-2a_1 x + 50x\right]_0^{10} = 0$$

$$-2a_1(10) + 50(10) = 0$$

$$-20a_1 = -500$$

$$a_1 = 25 \quad \rightarrow (4)$$

Trial function is $y = 25x(10-x)$

(iii) LEAST SQUARES METHOD

This method requires $I = \int_0^{10} R^2 \, dx$

It can be written as $\dfrac{\partial I}{\partial a_1} = \int_0^{10} R \dfrac{\partial R}{\partial a_1} \, dx \quad \rightarrow (5)$

We know that $R = -2a_1 + 50$

$$\frac{\partial R}{\partial a_1} = -2$$

Substitute R and $\frac{\partial R}{\partial a_1}$ in equation (5)

$$\frac{\partial I}{\partial a_1} = \int_0^{10} (-2a_1 + 50)(-2)\, dx$$

The requirement is $\frac{\partial I}{\partial a_1} = 0$

$$\int_0^{10} \left[4a_1 + (-100)\right] dx = 0$$

$$\int_0^{10} \left[4a_1\, dx - 100\, dx\right] = 0$$

$$\left[4a_1 x - 100x\right]_0^{10} = 0$$

$$40a_1 - 1000 = 0$$

$$a_1 = 25 \longrightarrow (6)$$

Trial function $y = 25x(10 - x)$

(iv) GALERKIN'S METHOD :

In this method trial function itself is considered as the weighting function w_i

$$\int_0^{10} w_i R\, dx = 0 \longrightarrow (7)$$

Trial function is $y = W_i = a_1 x(10-x)$

Substitute W_i and R values in equation (7)

$$\int_0^{10} a_1 x(10-x) \times (-2a_1+50)\, dx = 0$$

$$a_1 \int_0^{10} x(10-x) * (-2a_1+50)\, dx = 0$$

$$a_1 \int_0^{10} (10x - x^2)(-2a_1+50)\, dx = 0$$

$$a_1 \int_0^{10} \left[-20a_1 x + 500x + 2a_1 x^2 - 50x^2\right] dx = 0$$

$$a_1 \left[-20a_1 \frac{x^2}{2} + 500\frac{x^2}{2} + 2a_1 \frac{x^3}{3} - 50\frac{x^3}{3}\right]_0^{10} = 0$$

$$\frac{-20a_1}{2}\left[10^2 - 0\right] + \frac{500}{2}\left[10^2 - 0\right] + \frac{2a_1}{3}\left[10^3 - 0\right] - \frac{50}{3}\left[10^3 - 0\right] = 0$$

$$-10a_1(100) + 250(100) + \frac{2a_1}{3}(1000) - \frac{50}{3}(1000) = 0$$

$$-1000a_1 + 25000 + 666.66a_1 - 16666.66 = 0$$

$$-333.33a_1 = -8333.33$$

$$a_1 = 25 \rightarrow (8)$$

Trial function is $y = 25x(10-x)$

From equations (3), (4), (6) and (8) we know that the value of parameter a_1 is same for all the four methods.

4) The following differential equation is available for a physical phenomenon : $\frac{d^2y}{dx^2} - 10x^2 = 5$; $0 \le x \le 1$ The boundary conditions are : $y(0) = 0$, $y(1) = 0$. By using Galerkin's method of weighted residuals to find an approximate solution of the above differential equation and also compare with exact solution.

GIVEN:

DE $\frac{d^2y}{dx^2} - 10x^2 = 5 \rightarrow (1)$

Boundary conditions are $y(0) = 0$

$y(1) = 0$

TOFIND:

Approximate solution by using Galerkin's method

SOLUTION:

Trial function which satisfies the boundary condition is

$$y = a_1 x(x-1) = a_1 (x^2 - x)$$

$$\frac{dy}{dx} = a_1 (2x - 1)$$

$$\frac{d^2y}{dx^2} = 2a_1$$

Substitute $\frac{d^2y}{dx^2}$ value in given differential equation

Residual $R = 2a_1 - 10x^2 - 5$

In Galerkin's method $\int_0^1 w_i R \, dx = 0 \longrightarrow (2)$

Trial function $y = w_i = a_1 x (x-1)$

Substitute w_i and R values in equation (2)

$$\int_0^1 a_1 x (x-1) \times (2a_1 - 10x^2 - 5) \, dx = 0$$

$$a_1 \int_0^1 \left[2a_1 x^2 - 10x^4 - 5x^2 - 2a_1 x + 10x^3 + 5x \right] dx = 0$$

$$2a_1 \left[\frac{x^3}{3}\right]_0^1 - 10\left[\frac{x^5}{5}\right]_0^1 - 5\left[\frac{x^3}{3}\right]_0^1 - 2a_1\left[\frac{x^2}{2}\right]_0^1 + 10\left[\frac{x^4}{4}\right]_0^1 + 5\left[\frac{x^2}{2}\right]_0^1 = 0$$

$$\frac{2a_1}{3}(1-0) - \frac{10}{5}(1-0) - \frac{5}{3}(1-0) - \frac{2a_1}{2}(1-0) + \frac{10}{4}(1-0) + \frac{5}{2}(1-0) = 0$$

$$\frac{2a_1}{3} - 2 - 1.666 - a_1 + 2.5 + 2.5 = 0$$

$$0.666 a_1 - a_1 + 1.334 = 0$$

$$-0.334 a_1 = -1.334$$

$$a_1 = 4$$

Approximate solution is $y = 4x(x-1) \rightarrow (3)$

EXACT SOLUTION

$$\frac{d^2y}{dx^2} = 10x^2 + 5 \quad (\text{given})$$

$$\frac{dy}{dx} = \int \frac{d^2y}{dx^2} = \frac{10x^3}{3} + 5x + C_1$$

$$y = \int \frac{dy}{dx} = \frac{10x^4}{12} + \frac{5x^2}{2} + C_1 x + C_2$$

$$y = 0.833x^4 + 2.5x^2 + C_1 x + C_2 \rightarrow (4)$$

Apply boundary conditions when $x=0$, $y=0$

$$C_2 = 0$$

When $x=1$, $y=0$

$$0 = 0.833 + 2.5 + C_1 + C_2$$

$$C_1 = -3.333 \quad [C_2 = 0]$$

Substitute C_1 and C_2 values in equation (4)

$$y = 0.833x^4 + 2.5x^2 - 3.333x$$

Approximate solution $y = 4x(x-1)$

Exact solution $y = 0.833x^4 + 2.5x^2 - 3.333x$

RAYLEIGH - RITZ METHOD [VARIATIONAL APPROACH]

INTRODUCTION

* Rayleigh - Ritz method is a integral approach method which is useful for solving complex structural problems, encountered in finite element analysis. This method is possible only if a suitable functional is available, otherwise Galerkin's method of weighted residual is used. By using this method stiffness matrices and consistent load vector can be assembled easily. This method is mostly used for solving solid mechanics problems.
* The phrase "Variational methods" refers to methods that make use of variational principles, such as the principles of virtual work and the principle of minimum potential energy in solid and structural mechanics, to determine the approximate solutions of the problems.
* In Rayleigh - Ritz method for continuous system we deal with the following functional

$$\text{Potential energy } \pi = \int_{x_1}^{x_2} f(y, y', y'')\, dx$$

* In our terminology, a functional is an integral expression that implicitly contains the governing differential equations for a particular problem.

* Total potential energy of the structure is given by

$$\pi = \begin{Bmatrix} \text{Internal} \\ \text{Potential} \\ \text{Energy} \end{Bmatrix} - \begin{Bmatrix} \text{External} \\ \text{Potential} \\ \text{Energy} \end{Bmatrix}$$

= Strain energy − Work done by external forces

$$\pi = U - H$$

* In this method, the approximating functions must satisfy the boundary conditions and should be easy to use. Polynomials are generally used and sometimes sine and cosine terms are also used as approximating function.

* In general any exact function can be represented as a polynomial or trigonometric series with undetermined constants as shown below

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

or

$$y = a_1 \sin \frac{\pi x}{L} + a_2 \sin \frac{3\pi x}{L} + \ldots$$

The constants a_0, a_1, a_2 are unknowns known as Ritz parameters of the curve. When the parameters are infinite, the particular polynomial tends to match the exact value. So the accuracy depends upon the number of parameters chosen.

The following two conditions must be fulfilled by the approximating function.

① It should satisfy the geometric boundary conditions.

② The function must have atleast one Ritz parameter.

* In general, a Rayleigh - Ritz solution is rarely exact except in some special simple cases, but it becomes more accurate with the use of more parameters.

* This method can be understood clearly by solving the following examples

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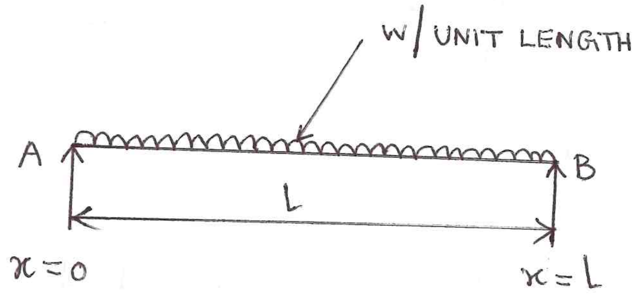
A simply supported beam subjected to uniformly distributed load over entire span. Determine the bending moment and deflection at midspan by using Rayleigh - Ritz method and compare with exact solutions.

GIVEN:



TOFIND:

① Deflection and Bending moment at midspan

② Compare with exact solutions.

SOLUTION:

The Fourier series for simply supported beam

$$y = \sum_{n=1,3}^{\infty} a \sin \frac{n\pi x}{L}$$ is the approximating function

Consider only two terms

Deflection $y = a_1 \sin \frac{\pi x}{L} + a_2 \sin \frac{3\pi x}{L} \rightarrow (1)$

where a_1, a_2 are Ritz parameters

We know that

Total PE of the beam $\pi = U - H \rightarrow (2)$

where, $U \rightarrow$ Strain energy

$H \rightarrow$ Work done by external force

Strain energy $U = \frac{EI}{2} \int_0^L \left(\frac{d^2y}{dx^2}\right)^2 dx \rightarrow (3)$

$$\frac{dy}{dx} = a_1 \cos \frac{\pi x}{L} \times \left(\frac{\pi}{L}\right) + a_2 \cos \frac{3\pi x}{L} \left(\frac{3\pi}{L}\right)$$

$$\frac{dy}{dx} = \frac{a_1 \pi}{L} \cos \frac{\pi x}{L} + \frac{a_2 3\pi}{L} \cos \frac{3\pi x}{L}$$

$$\frac{d^2y}{dx^2} = \frac{-a_1 \pi}{L} \sin \frac{\pi x}{L} \times \frac{\pi}{L} - a_2 \frac{3\pi}{L} \sin \frac{3\pi x}{L} \times \frac{3\pi}{L}$$

$$\frac{d^2y}{dx^2} = -\frac{\pi^2 a_1}{L^2} \sin \frac{\pi x}{L} - a_2 \frac{9\pi^2}{L^2} \sin \frac{3\pi x}{L} \rightarrow (4)$$

Substituting $\frac{d^2y}{dx^2}$ value in equation (3)

$$U = \frac{EI}{2}\int_0^L \left[-\frac{a_1\pi^2}{L^2}\sin\frac{\pi x}{L} - \frac{a_2\,9\pi^2}{L^2}\sin\frac{3\pi x}{L}\right]^2 dx$$

$$= \frac{EI}{2}\int_0^L \left[\frac{a_1\pi^2}{L^2}\sin\frac{\pi x}{L} + \frac{a_2\,9\pi^2}{L^2}\sin\frac{3\pi x}{L}\right]^2 dx$$

$$= \frac{EI}{2}\times\frac{\pi^4}{L^4}\int_0^L \left[a_1\sin\frac{\pi x}{L} + 9a_2\sin\frac{3\pi x}{L}\right]^2 dx$$

$$= \frac{EI}{2}\times\frac{\pi^4}{L^4}\int_0^L \left[a_1^2\sin^2\frac{\pi x}{L} + 81a_2^2\sin^2\frac{3\pi x}{L} + 2a_1\sin\frac{\pi x}{L}\,9a_2\sin\frac{3\pi x}{L}\right] dx$$

$$\left(\because (a+b)^2 = a^2+b^2+2ab\right)$$

$$U = \frac{EI}{2}\,\frac{\pi^4}{L^4}\int_0^L \left[a_1^2\sin^2\frac{\pi x}{L} + 81a_2^2\sin^2\frac{3\pi x}{L} + 18a_1a_2\sin\frac{\pi x}{L}\sin\frac{3\pi x}{L}\right] dx \quad \rightarrow (5)$$

$$\int_0^L a_1^2\sin^2\frac{\pi x}{L}\,dx = a_1^2\int_0^L \frac{1}{2}\left[1-\cos\frac{2\pi x}{L}\right]dx \qquad \left[\because \sin^2 x = \frac{1-\cos 2x}{2}\right]$$

$$= \frac{a_1^2}{2}\int_0^L \left[1-\cos\frac{2\pi x}{L}\right]dx$$

$$= \frac{a_1^2}{2}\left[\int_0^L dx - \int_0^L \cos\frac{2\pi x}{L}\,dx\right]$$

$$= \frac{a_1^2}{2}\left[(x)_0^L - \left(\frac{\sin\frac{2\pi x}{L}}{\frac{2\pi}{L}}\right)_0^L\right]$$

$$= \frac{a_1^2}{2}\left[L - 0 - \frac{L}{2\pi}\left(\sin\frac{2\pi L}{L} - \sin 0\right)\right]$$

$$= \frac{a_1^2}{2}\left[L - \frac{L}{2\pi}(0-0)\right] = \frac{a_1^2 L}{2} \qquad \left[\because \sin 2\pi = 0;\ \sin 0 = 0\right]$$

$$\int_0^L a_1^2 \sin^2\frac{\pi x}{L}\,dx = \frac{a_1^2 L}{2} \longrightarrow (6)$$

Similarly $$\int_0^L 81 a_2^2 \sin^2\frac{3\pi x}{L} = 81 a_2^2 \int_0^L \frac{1}{2}\left(1 - \cos\frac{6\pi x}{L}\right)dx$$

$$\left[\because \sin^2 x = \frac{1-\cos 2x}{2}\right]$$

$$= \frac{81 a_2^2}{2}\left[\int_0^L dx - \int_0^L \cos\frac{6\pi x}{L}\,dx\right]$$

$$= \frac{81 a_2^2}{2}\left[(x)_0^L - \left(\frac{\sin\frac{6\pi x}{L}}{\frac{6\pi}{L}}\right)_0^L\right]$$

$$= \frac{81 a_2^2}{2}\left[L - 0 - \frac{L}{6\pi}\left(\sin\frac{6\pi L}{L} - \sin 0\right)\right]$$

$$= \frac{81 a_2^2}{2}\left[L - \frac{L}{6\pi}\left(\sin 6\pi - \sin 0\right)\right]$$

$$= \frac{81 a_2^2}{2}(L - 0) \qquad \left[\because \sin 6\pi = 0;\ \sin 0 = 0\right]$$

$$\int_0^L 81 a_2^2 \sin^2\frac{3\pi x}{L} = \frac{81 a_2^2 L}{2} \longrightarrow (7)$$

$$\int_0^L 18 a_1 a_2 \sin\frac{\pi x}{L} \sin\frac{3\pi x}{L} = 18 a_1 a_2 \int_0^L \sin\frac{\pi x}{L} \sin\frac{3\pi x}{L}$$

$$= 18 a_1 a_2 \int_0^L \sin\frac{3\pi x}{L} \sin\frac{\pi x}{L}$$

$$= 18 a_1 a_2 \int_0^L \frac{1}{2}\left(\cos\frac{2\pi x}{L} - \cos\frac{4\pi x}{L}\right) dx$$

$$\left[\because \sin A \sin B = \frac{\cos(A-B) - \cos(A+B)}{2}\right]$$

$$= 18 a_1 a_2 \left[\int_0^L \cos\frac{2\pi x}{L}\, dx - \int_0^L \cos\frac{4\pi x}{L}\, dx\right]$$

$$= \frac{18 a_1 a_2}{2}\left[\int_0^L \cos\frac{2\pi x}{L}\, dx - \int_0^L \cos\frac{4\pi x}{L}\, dx\right]$$

$$= \frac{18 a_1 a_2}{2}\left[\left(\frac{\sin\frac{2\pi x}{L}}{\frac{2\pi}{L}}\right)_0^L - \left(\frac{\sin\frac{4\pi x}{L}}{\frac{4\pi}{L}}\right)_0^L\right]$$

$$\int_0^L 18 a_1 a_2 \sin\frac{\pi x}{L} \sin\frac{3\pi x}{L} = 9 a_1 a_2 [0 - 0] = 0 \rightarrow (8)$$

$$[\because \sin 2\pi = 0;\ \sin 4\pi = 0;\ \sin 0 = 0]$$

Substitute (6), (7) and (8) in equation (5)

$$U = \frac{EI}{2}\frac{\pi^4}{L^4}\left[\frac{a_1^2 L}{2} + \frac{81 a_2^2 L}{2} + 0\right]$$

$$U = \frac{EI \pi^4 L}{4 L^4}\left[a_1^2 + 81 a_2^2\right]$$

Strain energy $U = \frac{EI\pi^4}{4L^3}\left[a_1^2 + 81 a_2^2\right] \longrightarrow (9)$

We know that

Work done $H = \int_0^L w y \, dx = \int_0^L w \left[a_1 \sin\frac{\pi x}{L} + a_2 \sin\frac{3\pi x}{L}\right] dx$

$$= w \int_0^L \left[a_1 \sin\frac{\pi x}{L} + a_2 \sin\frac{3\pi x}{L}\right] dx$$

$$= w \left[a_1 \int_0^L \sin\frac{\pi x}{L}\, dx + a_2 \int_0^L \sin\frac{3\pi x}{L}\, dx\right]$$

$$= w \left[a_1 \left(\frac{-\cos\frac{\pi x}{L}}{\frac{\pi}{L}}\right)_0^L + a_2 \left(\frac{-\cos\frac{3\pi x}{L}}{\frac{3\pi}{L}}\right)_0^L\right]$$

$$= w \left[\frac{-a_1 L}{\pi}\left(\cos\frac{\pi x}{L}\right)_0^L - \frac{a_2 L}{3\pi}\left(\cos\frac{3\pi x}{L}\right)_0^L\right]$$

$$= w \left[\frac{-a_1 L}{\pi}\left[(-1) - 1\right] - \frac{a_2 L}{3\pi}(-1-1)\right]$$

$$= w \left[\frac{2a_1 L}{\pi} + \frac{2a_2 L}{3\pi}\right] \qquad \left[\because \cos 0 = 1;\ \cos\pi = -1;\ \cos 3\pi = -1\right]$$

$$= \frac{2wL}{\pi}\left[a_1 + \frac{a_2}{3}\right]$$

$$H = \frac{2wL}{\pi}\left[a_1 + \frac{a_2}{3}\right] \longrightarrow (10)$$

Substitute (9) and (10) values in equation (2)

$$\pi = U - H$$

$$\pi = \frac{EI\pi^4}{4l^3}\left[a_1^2 + 81a_2^2\right] - \frac{2wl}{\pi}\left[a_1 + \frac{a_2}{3}\right] \rightarrow (11)$$

For stationary value of π the following conditions must be satisfied

$$\frac{\partial \pi}{\partial a_1} = 0 \quad \text{and} \quad \frac{\partial \pi}{\partial a_2} = 0$$

$$\frac{\partial \pi}{\partial a_1} = \frac{EI\pi^4}{4l^3}(2a_1) - \frac{2wl}{\pi} = 0$$

$$\frac{EI\pi^4}{4l^3} \times 2a_1 = \frac{2wl}{\pi}$$

$$a_1 = \frac{4wl^4}{EI\pi^5}$$

$$\frac{\partial \pi}{\partial a_2} = \frac{EI\pi^4}{4l^3}(162a_2) - \frac{2wl}{\pi}\left(\frac{1}{3}\right) = 0$$

$$\frac{EI\pi^4}{4l^3}(162\,a_2) = \frac{2wl}{\pi}\left(\frac{1}{3}\right)$$

$$a_2 = \frac{2wl}{3\pi} \times \frac{4l^3}{162\,EI\pi^4} = \frac{4wl^4}{243\,EI\pi^5}$$

$$a_2 = \frac{4wl^4}{243\,EI\pi^5}$$

WKT $y = a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{3\pi x}{l}$

Substituting a_1 and a_2 values

$$y = \frac{4wL^4}{EI\pi^5} \sin\frac{\pi x}{L} + \frac{4wL^4}{243\,EI\pi^5} \sin\frac{3\pi x}{L} \longrightarrow (12)$$

We know that maximum deflection occurs at $x = \frac{L}{2}$

Substitute $x = \frac{L}{2}$ in equation (12)

$$y_{max} = \frac{4wL^4}{EI\pi^5} \sin\frac{\pi \times \frac{L}{2}}{L} + \frac{4wL^4}{243\,EI\pi^5} \sin\frac{3\pi \frac{L}{2}}{L}$$

$$y_{max} = \frac{4wL^4}{EI\pi^5} \sin\frac{\pi}{2} + \frac{4wL^4}{243\,EI\pi^5} \sin\frac{3\pi}{2}$$

$$y_{max} = \frac{4wL^4}{EI\pi^5} - \frac{4wL^4}{243\,EI\pi^5}$$

$$= \frac{4wL^4}{EI\pi^5}\left[1 - \frac{1}{243}\right] \qquad \left[\because \sin\frac{\pi}{2} = 1;\ \sin\frac{3\pi}{2} = -1\right]$$

$$= \frac{4wL^4}{EI\pi^5}(0.9958) = \frac{3.98\,wL^4}{EI\pi^5}$$

$$y_{max} = 0.0130\,\frac{wL^4}{EI} \longrightarrow (13)$$

For simply supported beam subjected to uniformly distributed load maximum deflection is

$$y_{max} = \frac{5}{384}\,\frac{wL^4}{EI}$$

$$\boxed{y_{max} = 0.0130\,\frac{wL^4}{EI}} \longrightarrow (14)$$

BENDING MOMENT AT MIDSPAN

Bending moment $M = EI \dfrac{d^2y}{dx^2}$

From equation (4) we know

$$\frac{d^2y}{dx^2} = -\left[\frac{a_1\pi^2}{l^2}\sin\frac{\pi x}{l} + a_2\frac{9\pi^2}{l^2}\sin\frac{3\pi x}{l}\right]$$

Substituting a_1 and a_2 values,

$$\frac{d^2y}{dx^2} = -\left[\frac{4wl^4}{EI\pi^5}\times\frac{\pi^2}{l^2}\sin\frac{\pi x}{l} + \frac{4wl^4}{243\,EI\pi^5}\times\frac{9\pi^2}{l^2}\sin\frac{3\pi x}{l}\right]$$

Maximum bending occurs at $x = \frac{l}{2}$

$$\frac{d^2y}{dx^2} = -\left[\frac{4wl^4}{EI\pi^5}\times\frac{\pi^2}{l^2}\sin\frac{\pi\frac{l}{2}}{l} + \frac{4wl^4}{243\,EI\pi^5}\times\frac{9\pi^2}{l^2}\sin\frac{3\pi\frac{l}{2}}{l}\right]$$

$$= -\left[\frac{4wl^4}{EI\pi^5}\times\frac{\pi^2}{l^2}\sin\frac{\pi}{2} + \frac{4wl^4}{243\,EI\pi^5}\times\frac{9\pi^2}{l^2}\sin\frac{3\pi}{2}\right]$$

$$= -\left[\frac{4wl^4}{EI\pi^5}\frac{\pi^2}{l^2}(1) + \frac{4wl^4}{243\,EI\pi^5}\times\frac{9\pi^2}{l^2}(-1)\right]$$

$$\left[\because \sin\frac{\pi}{2} = 1;\ \sin\frac{3\pi}{2} = -1\right]$$

$$= -\frac{4wl^2}{EI\pi^3} + \frac{36wl^2}{243\,EI\pi^3}$$

$$= -3.852\frac{wl^2}{EI\pi^3}$$

$$\frac{d^2y}{dx^2} = -0.124\frac{wL^2}{EI}$$

Substituting $\frac{d^2y}{dx^2}$ value in bending moment equation

$$\boxed{M_{centre} = EI \times -(0.124)\frac{wL^2}{EI} = -0.124wL^2} \rightarrow (16)$$

Negative sign indicates downward load

For simply supported beam subjected to uniformly distributed load maximum bending moment is

$$\boxed{M_{centre} = \frac{wL^2}{8} = 0.125\,wL^2} \rightarrow (17)$$

From equation (16) and (17) we know that exact solution and solution obtained by using Rayleigh – Ritz method are almost same. In order to get accurate result more terms in Fourier series should be taken.

UNIT II

ONE - DIMENSIONAL PROBLEMS

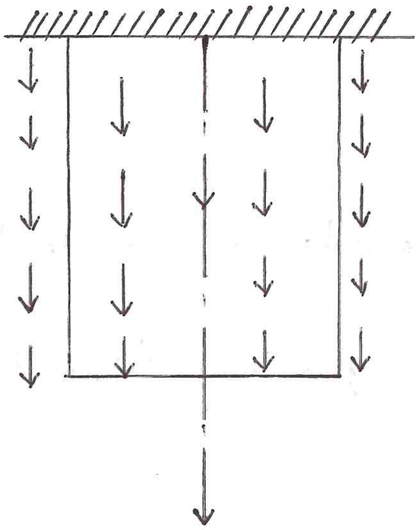
INTRODUCTION

Bar and beam elements are considered as one dimensional elements. These elements are often used to model trusses and frame structures.

A bar is a member which resist only axial loads, whereas a beam can resist axial, lateral and twisting loads. A truss is an assemblage of bars with pin joints and frame is an assemblage of beam elements.

In this chapter one dimensional elements and step-by-step procedure for the analysis of bars, trusses and beams are discussed. The total potential energy, stress-strain and strain-displacement relationships are used in developing the finite element method for a one dimensional problems. The basic one dimensional procedure is same for two and three dimensional problems.

STRESS, STRAIN, DISPLACEMENT AND LOADING



A BAR IS SUBJECTED TO LOADING

In one dimensional problems, stress (σ), strain (e), displacement (u) and loading depends only on the variable x. So, the vectors u, σ and e can be written as

$$u = u(x)$$

$$\sigma = \sigma(x)$$

$$e = e(x)$$

The stress-strain relationship is given by

$$\sigma = Ee$$

where $\sigma \rightarrow$ Stress, N/mm^2

$e \rightarrow$ Strain

$E \rightarrow$ Young's modulus, N/mm^2

The strain-displacement relationship is given by

$$e = \frac{du}{dx}$$

The differential volume can be written as

$$dV = A\,dx$$

There are three types of loading acts on the body. They are

(i) Body force (f)

(ii) Traction force (T)

(iii) Point load (P)

BODY FORCE (f)

A body force is a distributed force acting on every elemental volume of the body. Unit: Force per unit volume.

Example: Self weight due to gravity

TRACTION FORCE (T)

A traction force is a distributed force acting on the surface of the body. Unit: Force per unit area but for one dimensional problems, unit is force per unit length. Examples: Frictional resistance, viscous drag, surface shear etc

POINT LOAD (P)

Point load is a force acting at a particular point which causes displacement.

FINITE ELEMENT MODELLING

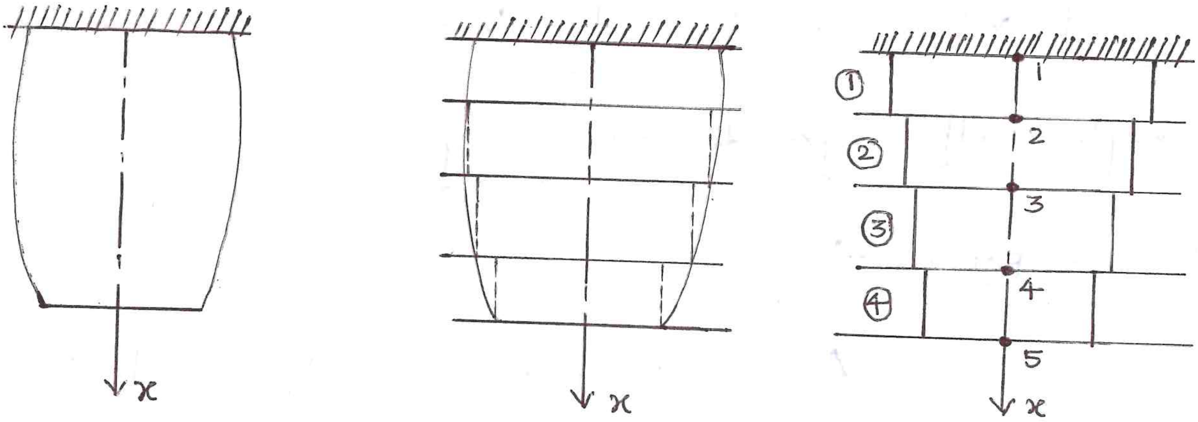
Finite element modelling consists of the following:

(i) Discretization of structure

(ii) Numbering of nodes.

(i) DISCRETIZATION

The art of subdividing a structure into a convenient number of smaller components is known as discretization.



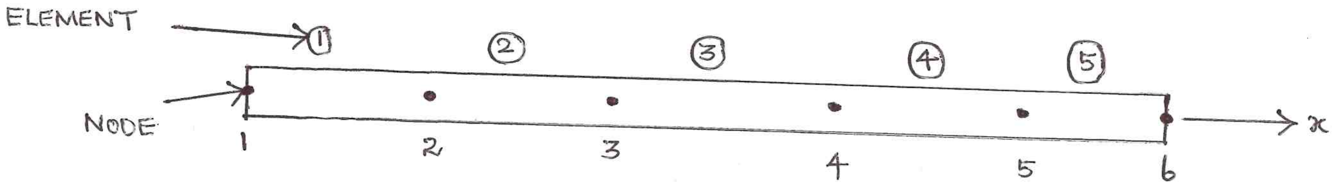
Consider a bar as shown in figure. The first step is to model the bar as a stepped shaft. Let us model the bar using 5 finite elements, each having a uniform cross section as shown in figure. Every elements connects two nodes. Four element, five node model element is shown in figure.

The element numbers are circled to distinguish them from node numbers. The cross-sectional area, traction forces and body forces are constant within each element. But, these are differ in magnitude from element to element. Better results are obtained by increasing the number of finite elements.

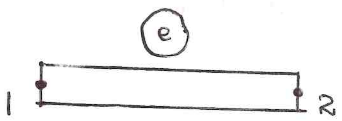
(ii) NUMBERING OF NODES

In one dimensional problem, each node is allowed to move only in $\pm x$ direction. So, each node has one degrees of freedom. Degrees of freedom is nothing but a nodal displacement.

A six node finite element model is shown in figure. It has six degrees of freedom. Load is considered as positive if it is acting along the $\pm x$ direction.



In the element connectivity table, the heading 1 and 2 refer to local node numbers of an element and the corresponding node numbers on the structure are called global numbers. Connectivity thus establishes the local-global correspondence.



LOCAL NUMBERING

ELEMENT (e)	NODES 1 2
1	1 2
2	2 3
3	3 4
4	4 5
5	5 6

LOCAL NUMBERS

GLOBAL NUMBERS

CONNECTIVITY TABLE

CO-ORDINATES

The co-ordinates are generally classified as follows:

(i) Global co-ordinates.

(ii) Local co-ordinates

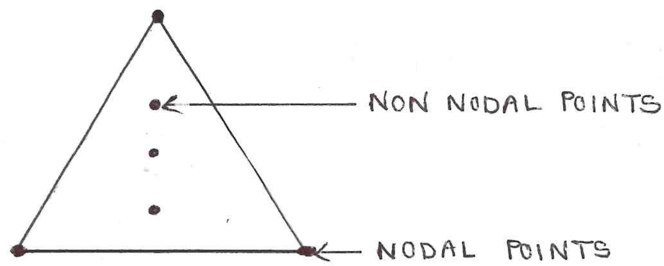
(iii) Natural co-ordinates

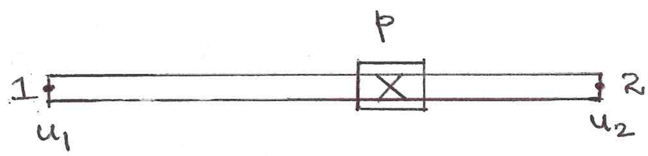
SHAPE FUNCTIONS:

INTRODUCTION:

If the values of the field variable are computed only at nodes, how are values obtained at other nodal points within a finite element? This is a most important point of finite element analysis.

The values of the field variable computed at the nodes are used to approximate the values at non-nodal points by interpolation of the nodal values.





In general, shape functions need to satisfy the following:

1. First derivatives should be finite within an element.
2. Displacement should be continuous across the element boundary.

The characteristics of shape function are:

1. The shape function has unit value at its own nodal point and zero value at other nodal points.
2. The sum of shape function is equal to one.
3. The shape functions for two dimensional elements are zero along each side that the node does not touch.
4. The shape functions are always polynomials of the same type as the original interpolation equations.

POLYNOMIAL SHAPE FUNCTIONS

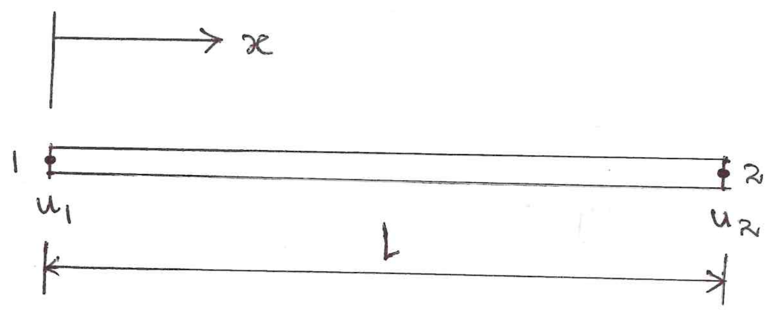
Polynomials are generally used as shape function due to the following reasons.

1. Differentiation and integration of polynomials are quite easy.
2. It is easy to formulate and computerize the finite element equations.
3. The accuracy of the results can be improved by increasing the order of the polynomial.

PROPERTIES OF STIFFNESS MATRIX

1. It is a symmetric matrix.
2. The sum of elements in any column must be equal to zero.
3. It is an unstable element. So the determinant is equal to zero.
4. The dimension of the global stiffness matrix [K] is N×N, where N is the number of nodes. This follows from the fact that each node has only one degree of freedom.
5. The diagonal coefficients are always positive and relatively large when compared to the off-diagonal values in the same row.

DERIVATION OF STIFFNESS MATRIX FOR ONE DIMENSIONAL BAR ELEMENT



A BAR ELEMENT WITH TWO NODES

Consider a one dimensional bar element with nodes 1 and 2 as shown in figure. Let u_1 and u_2 be the nodal displacement parameters or otherwise known as degrees of freedom.

We know that

$$\text{stiffness matrix } [K] = \int_V [B]^T [D] [B] \, dv$$

In one dimensional bar element,

Displacement function $u = N_1 u_1 + N_2 u_2$

where, $N_1 = \frac{l - x}{L}$

$N_2 = \frac{x}{l}$

we know that

Strain - Displacement matrix $[B] = \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} \end{bmatrix}$

$$[B] = \begin{bmatrix} \frac{-1}{l} & \frac{1}{l} \end{bmatrix}$$

$$[B]^T = \begin{Bmatrix} \frac{-1}{L} \\ \frac{1}{L} \end{Bmatrix}$$

In one dimensional problems $[D] = [E] = E$ = Young's modulus

Substitute $[B]$, $[B]^T$ and $[D]$ values in stiffness matrix equation.

[Limit is 0 to l]

$$[k] = \int_0^l \begin{Bmatrix} \frac{-1}{L} \\ \frac{1}{l} \end{Bmatrix} \times E \times \begin{bmatrix} \frac{-1}{l} & \frac{1}{L} \end{bmatrix} dv = \int_0^l \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} E \, dv$$

[$\because$ Matrix multiplication $(2 \times 1) \times (1 \times 2) = (2 \times 2)$]

$$= \int_0^l \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} E \, A \, dx$$

[$\because dv = A \, dx$]

$$= AE \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} \int_0^l dx = AE \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} [x]_0^l$$

$$= AE \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} (l-0) = AEl \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ -\frac{1}{l^2} & \frac{1}{l^2} \end{bmatrix}$$

$$= \frac{AEl}{l^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k] = \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The properties of a stiffness matrix are satisfied

1. It is symmetric
2. The sum of elements in any column is equal to zero

DERIVATION OF FINITE ELEMENT EQUATION FOR ONE DIMENSIONAL LINEAR BAR ELEMENT

We know that General force equation is

$$\{F\} = [k]\{u\}$$

where, $\{F\}$ = is a element force vector [Column matrix]

$[k]$ = is a stiffness matrix [Row matrix]

$\{u\}$ = is a nodal displacement [Column matrix]

For one dimensional bar element stiffness matrix $[k]$ is given by

$$[k] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

For two noded bar element

$$\{F\} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\{u\} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

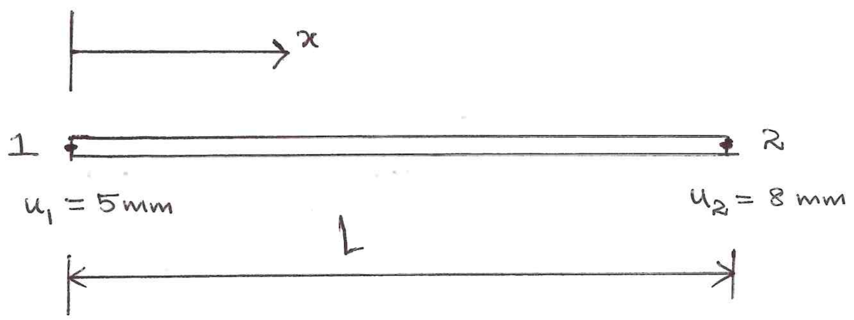
Substitute $[k]$ $\{F\}$ and $\{u\}$ values

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

This is the finite element equation for one dimensional two noded bar element

SOLVED PROBLEMS ON LINEAR BAR ELEMENTS

① A two noded truss element is shown in figure. The nodal displacements are $u_1 = 5$ mm and $u_2 = 8$ mm. Calculate the displacement at $x = \frac{L}{4}, \frac{L}{3}$ and $\frac{L}{2}$



GIVEN: Displacements $u_1 = 5$ mm

$u_2 = 8$ mm

TOFIND: Displacement u at $x = \frac{l}{4}, \frac{l}{3}$ and $\frac{l}{2}$

SOLUTION: Displacement function for two noded truss element is given by

$$u = N_1 u_1 + N_2 u_2$$

Where $N_1 = \frac{l - x}{l}$

$N_2 = \frac{x}{l}$

$$u = \left[\frac{l - x}{l}\right] u_1 + \left[\frac{x}{l}\right] u_2 \rightarrow (1)$$

Substitute $x = \frac{l}{4}$, $u_1 = 5$ and $u_2 = 8$ in equ (1)

$$u = \left[\frac{l - \frac{l}{4}}{l}\right] \times 5 + \left[\frac{\frac{l}{4}}{l}\right] \times 8$$

$$= \left[1 - \frac{1}{4}\right] \times 5 + \left[\frac{1}{4}\right] \times 8$$

$u = 5.75$ mm at $x = \frac{l}{4}$

Substitute $x = \frac{l}{3}$, $u_1 = 5$ mm and $u_2 = 8$ mm in equ (1)

$$u = \left[\frac{l - \frac{l}{3}}{l}\right] \times 5 + \left[\frac{\frac{l}{3}}{l}\right] \times 8$$

$$= \left[1-\frac{1}{3}\right]\times 5 + \left[\frac{1}{3}\right]\times 8$$

$u = 6$ mm at $x = \frac{L}{3}$

Substitute $x = \frac{L}{2}$, $u_1 = 5$ mm and $u_2 = 8$ mm in equation (1)

$$u = \left[\frac{L-\frac{L}{2}}{L}\right]\times 5 + \left[\frac{\frac{L}{2}}{L}\right]\times 8$$

$$= \left[1-\frac{1}{2}\right]\times 5 + \left[\frac{1}{2}\right]\times 8$$

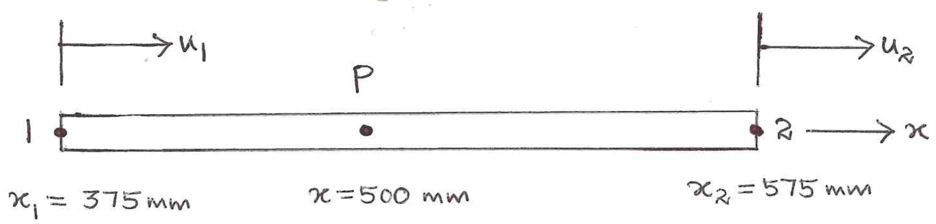
$u = 6.5$ mm at $x = \frac{L}{2}$

RESULT: $u = 5.75$ mm at $x = \frac{L}{4}$

$u = 6$ mm at $x = \frac{L}{3}$

$u = 6.5$ mm at $x = \frac{L}{2}$

Consider a bar as shown in figure. Cross-sectional area of the bar is 750 mm^2 and Young's modulus is 2×10^5 N/mm^2. If $u_1 = 0.5$ mm and $u_2 = 0.625$ mm, calculate the following: (i) Displacement at point, P (ii) Strain, e (iii) Stress, σ (iv) Element stiffness matrix [k] (v) Strain energy, U



GIVEN: Area $A = 750\ mm^2$

Young's modulus, $E = 2 \times 10^5\ N/mm^2$

Displacements $u_1 = 0.5$ mm

$u_2 = 0.625$ mm

Distance $x_1 = 375$ mm

$x_2 = 575$ mm

$x = 500$ mm

TO FIND: (i) Displacement at point P i.e., u

(ii) Strain, e

(iii) Stress, σ

(iv) Element stiffness matrix $[k]$

(v) Strain energy, U

SOLUTION: Actual length of the bar $L = x_2 - x_1 = 575 - 375 = 200$ mm

The distance between point 1 and point P is

$$x = 500 - 375 = 125 \text{ mm}$$

Displacement function for two noded bar element is

$$u = N_1 u_1 + N_2 u_2$$

Shape function $N_1 = \dfrac{L - x}{L}$; $N_2 = \dfrac{x}{L}$

$$N_1 = \frac{200-125}{200}$$

$$N_1 = 0.375$$

$$N_2 = \frac{125}{200}$$

$$N_2 = 0.625$$

Substitute N_1, N_2, u_1 and u_2 values in displacement equation

$$u = 0.375(0.5) + 0.625(0.625)$$

$$u = 0.5781 \text{ mm}$$

Strain $e = [B]\{u^*\}$

$$[B] = \begin{bmatrix} \frac{-1}{l} & \frac{1}{l} \end{bmatrix} = \begin{bmatrix} \frac{-1}{200} & \frac{1}{200} \end{bmatrix}$$

$$e = [B]\{u^*\} = \begin{bmatrix} \frac{-1}{200} & \frac{1}{200} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \begin{bmatrix} \frac{-1}{200} & \frac{1}{200} \end{bmatrix} \begin{Bmatrix} 0.5 \\ 0.625 \end{Bmatrix}$$

$$= \begin{bmatrix} \frac{-1}{200} \times 0.5 + \frac{1}{200} \times 0.625 \end{bmatrix}$$

$$e = 6.25 \times 10^{-4}$$

$$\sigma = Ee = 2 \times 10^5 \times 6.25 \times 10^{-4}$$

$$\sigma = 125 \text{ N/mm}^2$$

$$[k] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{750 \times 2 \times 10^5}{200}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k] = 7.5 \times 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$U = \frac{1}{2}\{u^*\}^T [k] \{u^*\}$$

$$= \frac{1}{2}\begin{bmatrix} u_1 & u_2 \end{bmatrix} \times 7.5 \times 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0.5 \\ 0.625 \end{Bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 0.5 & 0.625 \end{bmatrix} \times 7.5 \times 10^5 \begin{Bmatrix} 0.5 & -0.625 \\ -0.5 & +0.625 \end{Bmatrix}$$

$$= \frac{1}{2} \times 7.5 \times 10^5 \begin{bmatrix} 0.5 & 0.625 \end{bmatrix}\begin{Bmatrix} -0.125 \\ 0.125 \end{Bmatrix}$$

$$= \frac{1}{2} \times 7.5 \times 10^5 \left[0.5 \times (-0.125) + 0.625 \times 0.125\right]$$

$$U = 5859.37 \text{ Nmm}$$

RESULT:

(i) $u = 0.5781$ mm

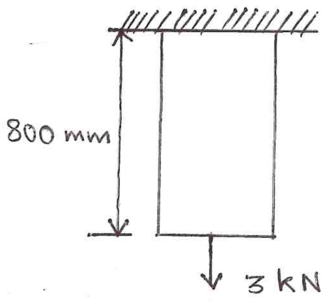
(ii) $e = 6.25 \times 10^{-4}$

(iii) $\sigma = 125$ N/mm^2

(iv) $[k] = 7.5 \times 10^5$

(v) $U = 5859.37$ Nmm

A steel bar of length 800 mm is subjected to an axial load of 3 kN as shown in figure. Find the elongation of the bar, neglecting self weight



Take $E = 2 \times 10^5$ N/mm², $A = 300$ mm²

GIVEN: Length $L = 800$ mm

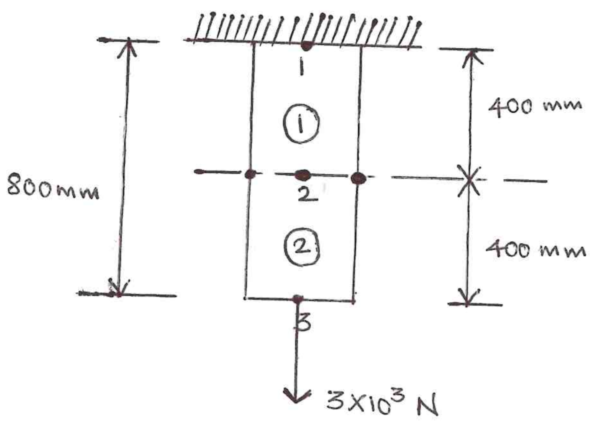
Load $F = 3$ kN $= 3 \times 10^3$ N

Young's modulus $E = 2 \times 10^5$ N/mm²

Area $A = 300$ mm²

TOFIND: Elongation u

SOLUTION: Divide the bar into two elements



Now the bar has 2 elements with 3 nodes

We can find the displacement at node 1, node 2 and node 3

Displacement at node 1 is u_1

node 2 is u_2

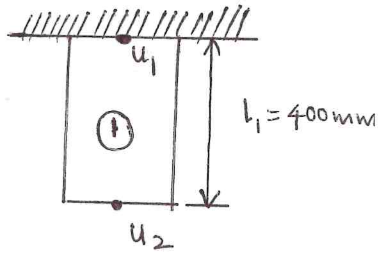
node 3 is u_3

For one dimensional two noded bar element the finite elements equation is

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

FOR ELEMENT 1: (Nodes 1, 2):

Finite element equation is



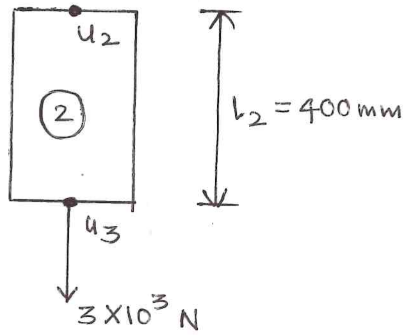
$$\frac{A_1 E}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\frac{300 \times 2 \times 10^5}{400}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$150 \times 10^3 \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \rightarrow (1)$$

FOR ELEMENT 2 : (Nodes 2,3) :

Finite element equation is,



$$\frac{A_2 E}{l_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

$$\frac{300 \times 2 \times 10^5}{400}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

$$150 \times 10^3 \begin{array}{c c} \begin{array}{cc} 2 & 3 \end{array} & \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & \begin{array}{c} 2 \\ 3 \end{array} \end{array} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} \longrightarrow (2)$$

Assemble the finite elements i.e., assemble the finite element equations (1) and (2)

$$150 \times 10^3 \begin{array}{c c} \begin{array}{ccc} 1 & 2 & 3 \end{array} & \\ \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1+1 & -1 \\ 0 & -1 & 1 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

$$\underbrace{150 \times 10^3 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}}_{[k]_{global}} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} \longrightarrow (3)$$

Applying Boundary Conditions.

$u_1 = 0$

$F_3 = 3 \times 10^3 N$

Self weight is neglected $F_1 = F_2 = 0$

$$150 \times 10^3 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 3 \times 10^3 \end{Bmatrix}$$

Here $u_1 = 0$ Neglect first row and first column of $[k]$ matrix.

Hence the final reduced equation is

$$150 \times 10^3 \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 3 \times 10^3 \end{Bmatrix}$$

$$150 \times 10^3 \left[2u_2 - u_3\right] = 0 \longrightarrow (4)$$

$$150 \times 10^3 \left[-u_2 + u_3\right] = 3 \times 10^3 \longrightarrow (5)$$

Solving equations (4) and (5)

$$150 \times 10^3 (u_2) = 3 \times 10^3$$

$$u_2 = 0.02 \text{ mm}$$

Substitute value of u_2 in equation (4)

$$150 \times 10^3 \left(2 \times 0.02 - u_3\right) = 0$$

$$2 \times 0.02 - u_3 = 0$$

$u_3 = 0.04\,mm$

VERIFICATION:

Total elongation $\delta L = \dfrac{pL}{AE} = \dfrac{3\times10^3 \times 800}{300 \times 2 \times 10^5} = 0.04\,mm$

RESULT:

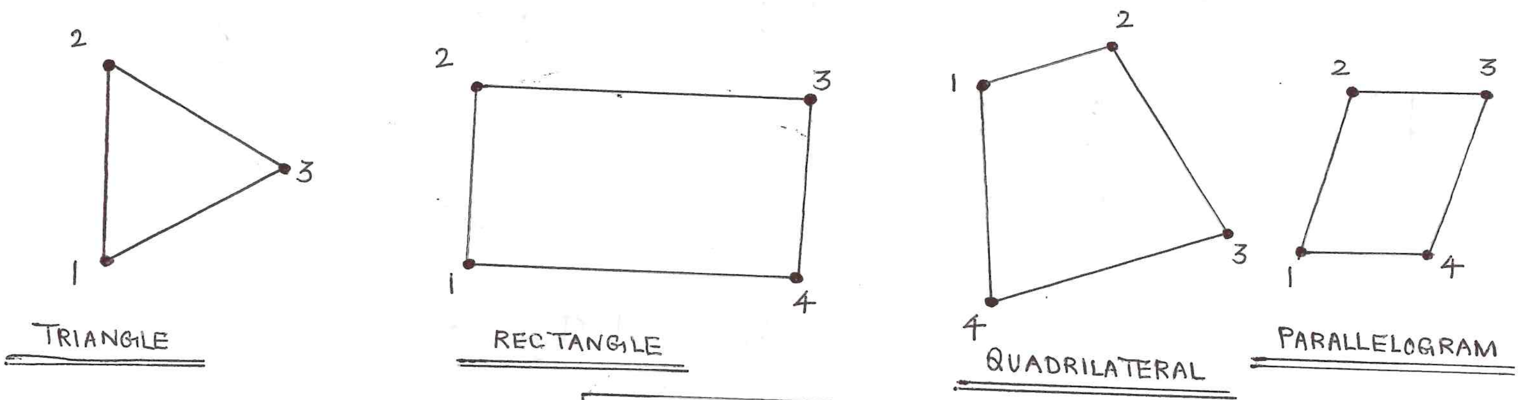
$u_1 = 0$; $u_2 = 0.02\,mm$; $u_3 = 0.04\,mm$

UNIT - III

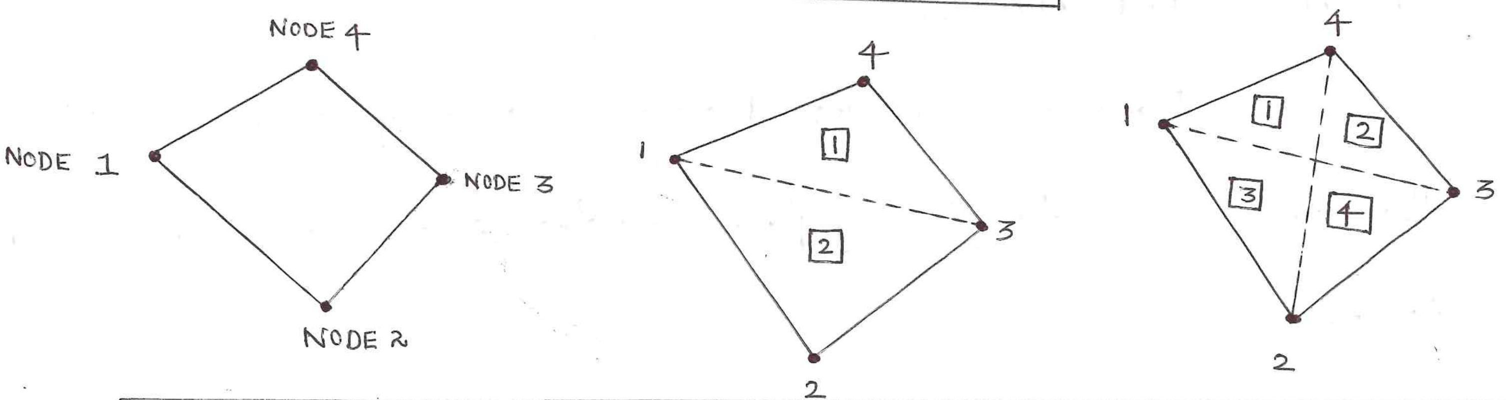
TWO DIMENSIONAL SCALAR VARIABLE PROBLEMS

INTRODUCTION

This chapter considers the two dimensional finite element. Two dimensional elements are defined by three or more nodes in a two dimensional plane (i.e., x, y plane). The basic element useful for two dimensional analysis is the triangular element. The simplest two dimensional elements have corner nodes as shown in figure. A quadrilateral (special forms of rectangle and parallelogram) element can be obtained by assembling two or four triangular elements. They are often used to model a wide range of Engineering problems.



TWO DIMENSIONAL ELEMENTS



A QUADRILATERAL ELEMENT AS AN ASSEMBLAGE OF TWO OR FOUR TRIANGULAR ELEMENTS

PLANE STRESS AND PLANE STRAIN

The two dimensional element is extremely important for the following two analysis

(i) Plane stress analysis

(ii) Plane strain analysis

(i) PLANE STRESS ANALYSIS:

Plane stress is defined to be a state of stress in which the normal stress (σ) and shear stress (τ) directed perpendicular to the plane are assumed to be zero.

Generally members that are thin and whose loads act only in the $x-y$ plane can be considered to be under plane stress.

Plates with holes and plates with fillets are coming under plane stress analysis problems.

Normal stress $\sigma_z = 0$

Shear stresses τ_{xz} and $\tau_{yz} = 0$

(ii) PLANE STRAIN ANALYSIS:

Plane strain is defined to be a state of strain in which the strain normal to the xy plane and the shear strains are assumed to be zero.

Dams and pipes subjected to loads that remain constant over their lengths are coming under plane strain analysis problems.

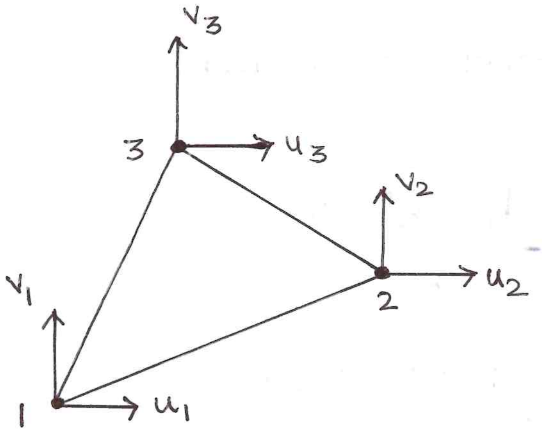
FINITE ELEMENT MODELLING:

Finite element modelling consists of the following:

(i) Discretization of structure

(ii) Numbering of nodes

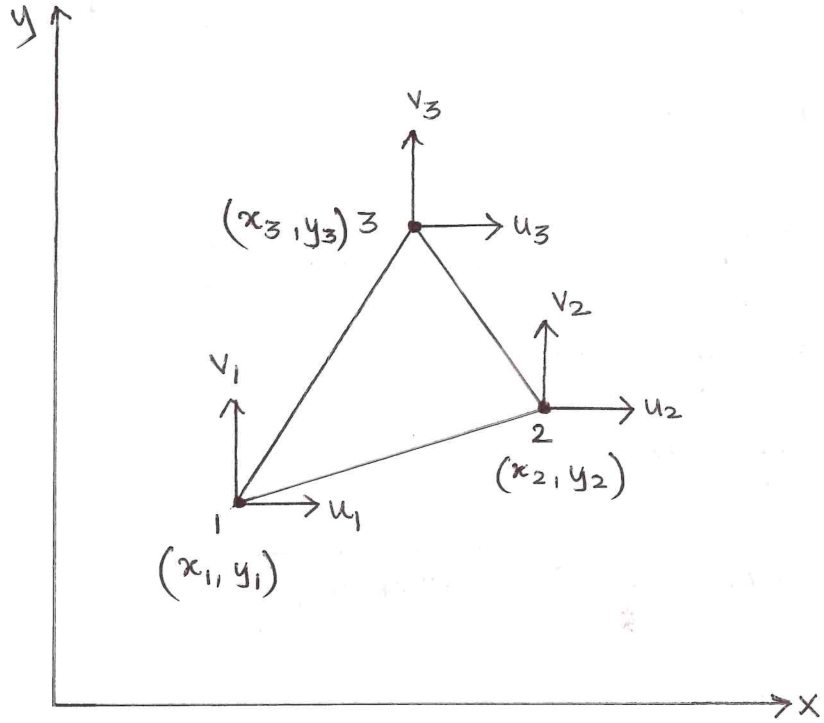
CONSTANT STRAIN TRIANGULAR (CST) ELEMENT:



CONSTANT STRAIN TRIANGULAR ELEMENT

A three noded triangular element is known as constant strain triangular (CST) element which is shown in figure. It has six unknown displacement degrees of freedom (u_1, v_1, u_2, v_2, u_3, v_3). The element is called CST because it has a constant strain throughout it.

SHAPE FUNCTION FOR THE CST ELEMENT



THREE NODED CST ELEMENT

Shape function $N_1 = \dfrac{p_1 + q_1 x + r_1 y}{2A}$

$$N_2 = \frac{p_2 + q_2 x + r_2 y}{2A}$$

$$N_3 = \frac{p_3 + q_3 x + r_3 y}{2A}$$

Displacement function $u = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$

FORMULAE USED:

① For constant strain triangle (CST) element

Shape function $N_1 + N_2 + N_3 = 1$

Co-ordinate $x = N_1 x_1 + N_2 x_2 + N_3 x_3$

Co-ordinate $y = N_1 y_1 + N_2 y_2 + N_3 y_3$

OR

Co-ordinate $x = (x_1 - x_3) N_1 + (x_2 - x_3) N_2 + x_3$

Co-ordinate $y = (y_1 - y_3) N_1 + (y_2 - y_3) N_2 + y_3$

② Area of the triangular element $A = \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$

③ Strain-Displacement matrix for CST element is

$$[B] = \frac{1}{2A} \begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix}$$

where $q_1 = y_2 - y_3$; $q_2 = y_3 - y_1$; $q_3 = y_1 - y_2$

$r_1 = x_3 - x_2$; $r_2 = x_1 - x_3$; $r_3 = x_2 - x_1$

(4) Stress - Strain relationship matrix for plane stress problem

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

where $\nu \rightarrow$ Poisson's ratio

$E \rightarrow$ Young's modulus

(5) Stress - Strain relationship matrix for plane strain problem,

$$[D] = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

(6) Element stiffness matrix for CST element

$$[K] = [B]^T [D] [B] At$$

(7) Element stress $[\sigma] = [D][B]\{u\}$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [D][B]\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

where $\sigma_x, \sigma_y \rightarrow$ Normal stresses

$\tau_{xy} \rightarrow$ Shear stress

$u, v \rightarrow$ Nodal displacement

8) Maximum normal stress $\sigma_{max} = \sigma_1 = \dfrac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$

Minimum normal stress $\sigma_{min} = \sigma_2 = \dfrac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$

9) Principal angle $\tan 2\theta_p = \dfrac{2\tau_{xy}}{\sigma_x - \sigma_y}$

10) Element strain $\{e\} = [B]\{u\} = [B]\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$

11) Temperature effects

$$\left.\begin{array}{l}\text{Initial strain } \{e_0\} \\ \text{(For plane stress problems)}\end{array}\right\} = \begin{Bmatrix} \alpha \Delta T \\ \alpha \Delta T \\ 0 \end{Bmatrix}$$

$$\left.\begin{array}{l}\text{Initial strain } \{e_0\} \\ \text{(For plane strain problems)}\end{array}\right\} = (1+\nu)\begin{Bmatrix} \alpha \Delta T \\ \alpha \Delta T \\ 0 \end{Bmatrix}$$

where $\alpha \rightarrow$ Coefficient of thermal expansion

$\nu \rightarrow$ Poisson's ratio

12) Element temperature force $\{F\} = [B]^T [D] \{e_0\}\, t A$

PROBLEMS ON CST ELEMENTS:

① Determine the shape functions N_1, N_2 and N_3 at the interior point P for the triangular element shown in figure

GIVEN DATA:

$x_1 = 2 \quad , y_1 = 3$

$x_2 = 7 \quad , \quad y_2 = 4$

$x_3 = 4 \quad , \quad y_3 = 7$

$x = 3.5 \quad , \quad y = 5$

TOFIND:

Shape function N_1, N_2 and N_3 at the interior point P

SOLUTION:

$$x = (x_1 - x_3) N_1 + (x_2 - x_3) N_2 + x_3 \rightarrow ①$$

$$y = (y_1 - y_3) N_1 + (y_2 - y_3) N_2 + y_3 \rightarrow ②$$

Sub the values

$$① \Rightarrow \quad 3.5 = (2 - 4) N_1 + (7 - 4) N_2 + 4 \rightarrow ③$$

$$② \Rightarrow \quad 5 = (3 - 7) N_1 + (4 - 7) N_2 + 7 \rightarrow ④$$

Eqn ③ becomes

$$3.5 = -2N_1 + 3N_2 + 4$$

$$-0.5 = -2N_1 + 3N_2$$

$$2N_1 - 3N_2 = 0.5 \rightarrow ⑤$$

Eqn 4 becomes

$$5 = -4N_1 - 3N_2 + 7$$

$$-2 = -4N_1 - 3N_2 \rightarrow 6$$

Solving equation 5 and 6

$$2N_1 - 3N_2 = 0.5$$

$$4N_1 + 3N_2 = 2$$

$$6N_1 = 2.5$$

$$N_1 = 0.4166$$

Sub N_1 value in eqn 5 or eqn 6

$$2N_1 - 3N_2 = 0.5$$

$$2 \times 0.4166 - 3N_2 = 0.5$$

$$N_2 = 0.1111$$

We know that $N_1 + N_2 + N_3 = 1$

$$0.4166 + 0.1111 + N_3 = 1$$

$$N_3 = 0.4723$$

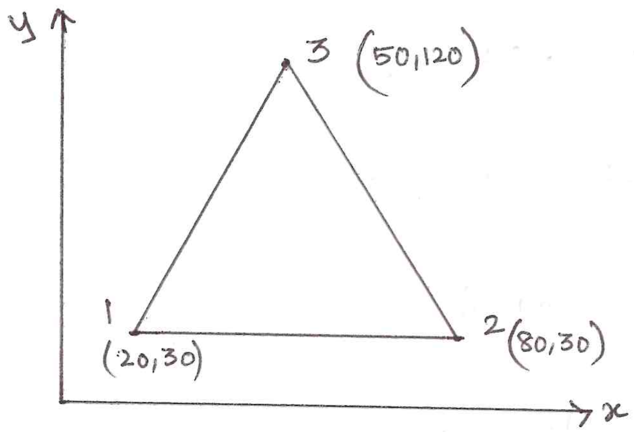
Shape functions at the interior point P

$$N_1 = 0.4166$$

$$N_2 = 0.1111$$

$$N_3 = 0.4723$$

② Calculate the stiffness matrix for the element shown in figure



The co-ordinates are given in units of millimeters. Assume plane stress condition. Take $E = 2.1 \times 10^5 \ N/mm^2$, $\nu = 0.25$, $t = 10$ mm

GIVEN DATA:

$x_1 = 10, \ y_1 = 7.5$

$x_2 = 15, \ y_2 = 5$

$x_3 = 15, \ y_3 = 10$

Young's modulus $E = 2.1 \times 10^5 \ N/mm^2$

Poisson's ratio $\nu = 0.25$

Thickness $t = 10$ mm

Assume plane stress condition

TOFIND:

Stiffness matrix $[k]$

SOLUTION:

Stiffness matrix $[k] = [B]^T [D] [B] A t$

$$A = \text{Area of the element} = \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$= \frac{1}{2} \begin{vmatrix} 1 & 10 & 7.5 \\ 1 & 15 & 5 \\ 1 & 15 & 10 \end{vmatrix}$$

$$= \frac{1}{2} \left[1(15 \times 10 - 15 \times 5) - 10(1 \times 10 - 1 \times 5) + 7.5(1 \times 15 - 1 \times 15) \right]$$

$$= \frac{1}{2}(75 - 50)$$

$$A = 12.5 \text{ mm}^2$$

Strain - Displacement matrix

$$[B] = \frac{1}{2A} \begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix}$$

$$q_1 = y_2 - y_3 = 5 - 10 = -5$$

$$q_2 = y_3 - y_1 = 10 - 7.5 = 2.5$$

$$q_3 = y_1 - y_2 = 7.5 - 5 = 2.5$$

$$r_1 = x_3 - x_2 = 15 - 15 = 0$$

$$r_2 = x_1 - x_3 = 10 - 15 = -5$$

$$r_3 = x_2 - x_1 = 15 - 10 = 5$$

$$[B] = \frac{1}{2A} \begin{bmatrix} -5 & 0 & 2.5 & 0 & 2.5 & 0 \\ 0 & 0 & 0 & -5 & 0 & 5 \\ 0 & -5 & -5 & 2.5 & 5 & 2.5 \end{bmatrix}$$

$$= \frac{1}{2 \times 12.5} \begin{bmatrix} -5 & 0 & 2.5 & 0 & 2.5 & 0 \\ 0 & 0 & 0 & -5 & 0 & 5 \\ 0 & -5 & -5 & 2.5 & 5 & 2.5 \end{bmatrix}$$

$$= \frac{2.5}{2 \times 12.5} \begin{bmatrix} -2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 1 \\ 0 & -2 & -2 & 1 & 2 & 1 \end{bmatrix}$$

$$[B] = 0.1 \begin{bmatrix} -2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 1 \\ 0 & -2 & -2 & 1 & 2 & 1 \end{bmatrix}$$

Stress - strain relationship matrix [D] for plane stress problem

$$[D] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

$$= \frac{2.1 \times 10^5}{1 - 0.25^2} \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \frac{1-0.25}{2} \end{bmatrix}$$

$$= \frac{2.1 \times 10^5}{0.9375} \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}$$

$$= \frac{2.1 \times 10^5 \times 0.25}{0.9375} \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

$$[D] = 5.6 \times 10^3 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

$$[D][B] = 5.6 \times 10^3 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} \times 0.1 \begin{bmatrix} -2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 2 \\ 0 & -2 & -2 & 1 & 2 & 1 \end{bmatrix}$$

$$[D][B] = 5.6 \times 10^3 \begin{bmatrix} -8 & 0 & 4 & -2 & 4 & 2 \\ -2 & 0 & 1 & -8 & 1 & 8 \\ 0 & -3 & -3 & 1.5 & 3 & 1.5 \end{bmatrix}$$

$$[B] = 0.1 \begin{bmatrix} -2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 2 \\ 0 & -2 & -2 & 1 & 2 & 1 \end{bmatrix}$$

$$[B]^T = 0.1 \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & -2 \\ 1 & 0 & -2 \\ 0 & -2 & 1 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

$$[B]^T[D][B] = 0.1 \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & -2 \\ 1 & 0 & -2 \\ 0 & -2 & 1 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix} \times 5.6 \times 10^3 \begin{bmatrix} -8 & 0 & 4 & -2 & 4 & 2 \\ -2 & 0 & 1 & -8 & 1 & 8 \\ 0 & -3 & -3 & 1.5 & 3 & 1.5 \end{bmatrix}$$

$$[B]^T[D][B] = 560 \begin{bmatrix} 16 & 0 & -8 & 4 & -8 & -4 \\ 0 & 6 & 6 & -3 & -6 & -3 \\ -8 & 6 & 10 & -5 & -2 & -1 \\ 4 & -3 & -5 & 17.5 & 1 & -14.5 \\ -8 & -6 & -2 & 1 & 10 & 5 \\ -4 & -3 & -1 & -14.5 & 5 & 17.5 \end{bmatrix}$$

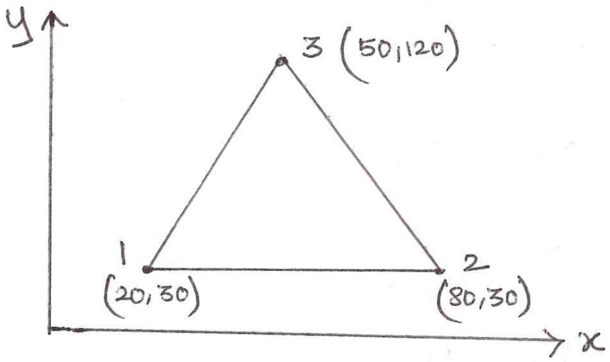
$$[K] = 560 \begin{bmatrix} 16 & 0 & -8 & 4 & -8 & -4 \\ 0 & 6 & 6 & -3 & -6 & -3 \\ -8 & 6 & 10 & -5 & -2 & -1 \\ 4 & -3 & -5 & 17.5 & 1 & -14.5 \\ -8 & -6 & -2 & 1 & 10 & 5 \\ -4 & -3 & -1 & -14.5 & 5 & 17.5 \end{bmatrix} \times 12.5 \times 10$$

$$[K] = 70 \times 10^3 \begin{bmatrix} 16 & 0 & -8 & 4 & -8 & -4 \\ 0 & 6 & 6 & -3 & -6 & -3 \\ -8 & 6 & 10 & -5 & -2 & -1 \\ 4 & -3 & -5 & 17.5 & 1 & -14.5 \\ -8 & -6 & -2 & 1 & 10 & 5 \\ -4 & -3 & -1 & -14.5 & 5 & 17.5 \end{bmatrix}$$

(3) For the plane stress element shown in figure the nodal displacements are $u_1 = 2.0$ mm $v_1 = 1.0$ mm

$u_2 = 0.5$ mm $v_2 = 0.0$ mm

$u_3 = 3.0$ mm $v_3 = 1.0$ mm



Determine the element stresses σ_x, σ_y, τ_{xy}, σ_1 and σ_2 and the principal angle θ_p. Let E = 210 GPa, $\nu = 0.25$ and t = 10 mm. All co-ordinates are in millimeters.

GIVEN DATA:

$u_1 = 2.0$ mm $v_1 = 1.0$ mm

$u_2 = 0.5$ mm $v_2 = 0.0$ mm

$u_3 = 3.0$ mm $v_3 = 1.0$ mm

$x_1 = 20$ mm $y_1 = 30$ mm

$x_2 = 80$ mm $y_2 = 30$ mm

$x_3 = 50$ mm $y_3 = 120$ mm

Young's modulus $E = 210\ GPa = 210 \times 10^9\ Pa$

$= 210 \times 10^9\ N/m^2$

$E = 2.1 \times 10^5\ N/mm^2$

Poisson's ratio $\nu = 0.25$

Thickness $t = 10$ mm

TOFIND:

① Element stresses

ⓐ Normal stress, σ_x

ⓑ Normal stress, σ_y

ⓒ Shear stress, τ_{xy}

ⓓ Maximum normal stress, σ_1

ⓔ Minimum normal stress, σ_2

② Principal angle, θ_p

SOLUTION:

$$\text{Area } A = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 20 & 30 \\ 1 & 80 & 30 \\ 1 & 50 & 120 \end{vmatrix}$$

$$= \frac{1}{2}\left[1\,(80\times120 - 50\times30) - 20\,(1\times120 - 1\times30) + 30\,(1\times50 - 1\times80)\right]$$

$$A = 2700 \text{ mm}^2$$

$$[B] = \frac{1}{2A}\begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix}$$

$$q_1 = y_2 - y_3 = 30 - 120 = -90$$

$$q_2 = y_3 - y_1 = 120 - 30 = 90$$

$$q_3 = y_1 - y_2 = 30 - 30 = 0$$

$$r_1 = x_3 - x_2 = 50 - 80 = -30$$

$$r_2 = x_1 - x_3 = 20 - 50 = -30$$

$$r_3 = x_2 - x_1 = 80 - 20 = 60$$

$$[B] = \frac{1}{2 \times 2700} \begin{bmatrix} -90 & 0 & 90 & 0 & 0 & 0 \\ 0 & -30 & 0 & -30 & 0 & 60 \\ -30 & -90 & -30 & 90 & 60 & 0 \end{bmatrix}$$

$$= \frac{30}{2 \times 2700} \begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -3 & -1 & 3 & 2 & 0 \end{bmatrix}$$

$$[B] = 5.555 \times 10^{-3} \begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -3 & -1 & 3 & 2 & 0 \end{bmatrix}$$

$$[D] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

$$= \frac{2.1 \times 10^5}{1 - 0.25^2} \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \frac{1-0.25}{2} \end{bmatrix}$$

$$= \frac{2.1\times10^5 \times 0.25}{0.9375}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

$$[D] = 56\times10^3 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

$$[D][B] = 56\times10^3 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} \times 5.555\times10^{-3} \begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -3 & -1 & 3 & 2 & 0 \end{bmatrix}$$

$$[D][B] = 311.08 \begin{bmatrix} -12 & -1 & 12 & -1 & 0 & 2 \\ -3 & -4 & 3 & -4 & 0 & 8 \\ -1.5 & -4.5 & -1.5 & 4.5 & 3 & 0 \end{bmatrix}$$

Stress $\{\sigma\} = [D][B]\{u\}$

$$= [D][B] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$= 311.08 \begin{bmatrix} -12 & -1 & 12 & -1 & 0 & 2 \\ -3 & -4 & 3 & -4 & 0 & 8 \\ -1.5 & -4.5 & -1.5 & 4.5 & 3 & 0 \end{bmatrix} \times \begin{Bmatrix} 2 \\ 1 \\ 0.5 \\ 0.0 \\ 3.0 \\ 1.0 \end{Bmatrix}$$

$$\sigma = 311.08 \begin{Bmatrix} -17 \\ -0.5 \\ 0.75 \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} -5288.36 \\ -155.54 \\ 233.31 \end{Bmatrix}$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \frac{-5288.36 - 155.54}{2} + \sqrt{\left(\frac{-5288.36 + 155.54}{2}\right)^2 + 233.31^2}$$

$$\sigma_1 = -144.956 \ N/mm^2$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \frac{-5288.36 - 155.54}{2} - \sqrt{\left(\frac{-5288.36 + 155.54}{2}\right)^2 + 233.31^2}$$

$$\sigma_2 = -5298.9 \ N/mm^2$$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$$2\theta_p = \tan^{-1}\left[\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right]$$

$$= \tan^{-1}\left[\frac{2 \times 233.31}{-5288.36 + 155.54}\right]$$

$$= -5.194^\circ$$

$$\theta_p = -2.59^\circ$$

(1) Element stresses

(a) Normal stress $\sigma_x = -5288.36 \ N/mm^2$

(b) Normal stress $\sigma_y = -155.54 \ N/mm^2$

(c) Shear stress $\tau_{xy} = 233.31 \ N/mm^2$

(d) Maximum normal stress $\sigma_1 = -144.956 \ N/mm^2$

(e) Minimum normal stress $\sigma_2 = -5298.9 \ N/mm^2$

(2) Principal angle $\theta_p = -2.59^\circ$

UNIT IV

TWO DIMENSIONAL VECTOR VARIABLE PROBLEMS

AXISYMMETRIC ELEMENTS

INTRODUCTION:

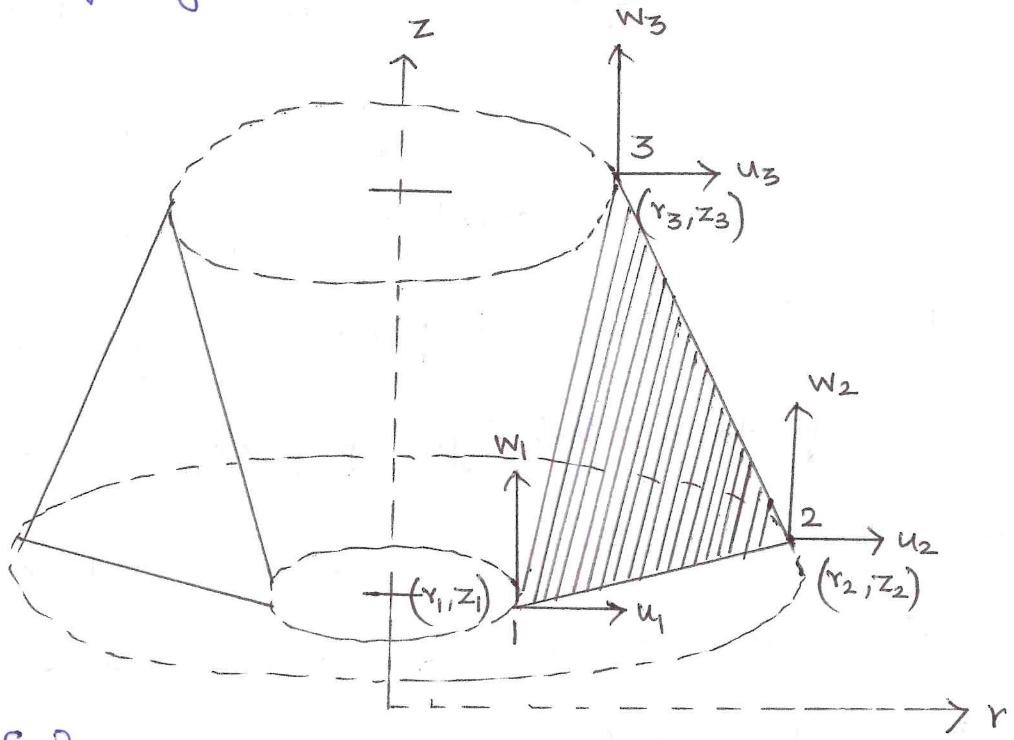
In previous chapters we have been concerned with one dimensional elements and two dimensional elements. In this chapter, we consider a special two dimensional element called the axisymmetric element.

Many three dimensional problems in engineering exhibit symmetry about an axis of rotation. Such types of problems are known as axisymmetric problems. These problems can be solved by using two dimensional finite elements. These elements are most conveniently described in cylindrical (r, θ, z) co-ordinates. The required conditions for a problem to be axisymmetric are as follows:

1. The problem domain must be symmetric about the axis of revolution, which is conveniently taken as the z-axis
2. All boundary conditions must be symmetric about the axis of revolution
3. All loading conditions must be symmetric about the axis of revolution.

An axisymmetric solid is generated by revolving a plane figure about an axis in the plane.

Finite elements for axisymmetric solids are pictured as triangular element or quadrilateral element. But these shapes are actually cross-sections of ring elements.



$$\{e\} = [B]\{u\}$$

$[B]$ = Strain - Displacement matrix.

$$= \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1}{r}+\beta_1+\frac{\gamma_1 Z}{r} & 0 & \frac{\alpha_2}{r}+\beta_2+\frac{\gamma_2 Z}{r} & 0 & \frac{\alpha_3}{r}+\beta_3+\frac{\gamma_3 Z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

where $\alpha_1 = r_2 z_3 - r_3 z_2$

$$\alpha_2 = r_3 z_1 - r_1 z_3$$

$$\alpha_3 = r_1 z_2 - r_2 z_1$$

$$\beta_1 = z_2 - z_3$$

$$\beta_2 = z_3 - z_1$$

$$\beta_3 = z_1 - z_2$$

$$\gamma_1 = r_3 - r_2$$

$$\gamma_2 = r_1 - r_3$$

$$\gamma_3 = r_2 - r_1$$

$[D]$ = Stress - strain relationship matrix

$$= \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

$[k]$ = Stiffness matrix = $2\pi r A [B]^T [D] [B]$

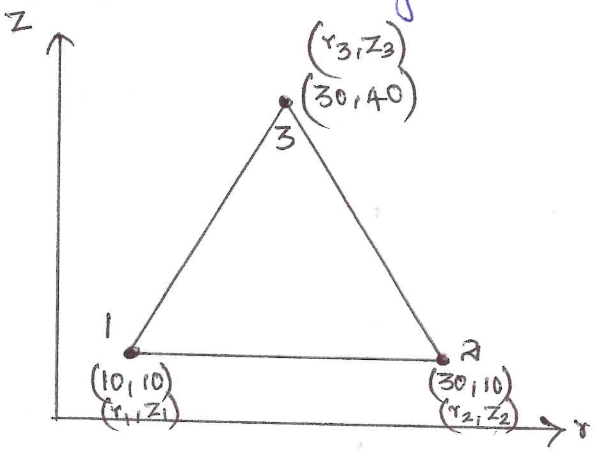
PROBLEMS ON AXISYMMETRIC ELEMENT

① The nodal co-ordinates for an axisymmetric triangular element are given below

$r_1 = 10$ mm $z_1 = 10$ mm

$r_2 = 30$ mm $z_2 = 10$ mm

$r_3 = 30$ mm $z_3 = 40$ mm



Evaluate [B] matrix for that element

GIVEN DATA:

$r_1 = 10$ mm $z_1 = 10$ mm

$r_2 = 30$ mm $z_2 = 10$ mm

$r_3 = 30$ mm $z_3 = 40$ mm

TO FIND:

[B]

SOLUTION:

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1}{r}+\beta_1+\frac{\gamma_1 z}{r} & 0 & \frac{\alpha_2}{r}+\beta_2+\frac{\gamma_2 z}{r} & 0 & \frac{\alpha_3}{r}+\beta_3+\frac{\gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

where A = Area of the triangular element

$$= \frac{1}{2}\begin{vmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{vmatrix}$$

$$= \frac{1}{2}\left[\left(r_2 z_3 - r_3 z_2\right) - r_1\left(z_3 - z_2\right) + z_1\left(r_3 - r_2\right)\right]$$

$$= \frac{1}{2}\left[(30 \times 40) - (30 \times 10) - 10\,(40-10) + 10\,(30-30)\right]$$

$$= \frac{1}{2} \times 600$$

$$A = 300 \text{ mm}^2$$

$$r = \frac{r_1 + r_2 + r_3}{3} = \frac{10 + 30 + 30}{3} = 23.334 \text{ mm}$$

$$z = \frac{z_1 + z_2 + z_3}{3} = \frac{10 + 10 + 40}{3} = 20 \text{ mm}$$

$$\alpha_1 = r_2 z_3 - r_3 z_2 = (30 \times 40) - (30 \times 10) = 900 \text{ mm}^2$$

$$\alpha_2 = r_3 z_1 - r_1 z_3 = (30 \times 10) - (10 \times 40) = -100 \text{ mm}^2$$

$$\alpha_3 = r_1 z_2 - r_2 z_1 = (10 \times 10) - (30 \times 10) = -200 \text{ mm}^2$$

$$\beta_1 = z_2 - z_3 = 10 - 40 = -30 \text{ mm}$$

$$\beta_2 = z_3 - z_1 = 40 - 10 = 30 \text{ mm}$$

$$\beta_3 = z_1 - z_2 = 10 - 10 = 0$$

$$\gamma_1 = r_3 - r_2 = 30 - 30 = 0$$

$$\gamma_2 = r_1 - r_3 = 10 - 30 = -20 \text{ mm}$$

$$\gamma_3 = r_2 - r_1 = 30 - 10 = 20 \text{ mm}$$

$$\frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} = \frac{900}{23.334} + (-30) + \frac{0 \times 20}{23.334} = 8.571 \text{ mm}$$

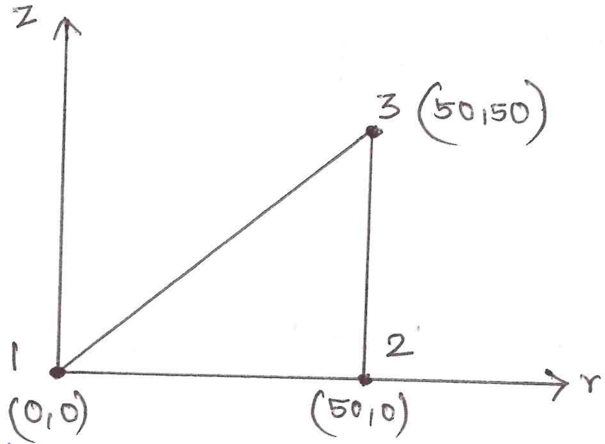
$$\frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} = \frac{-100}{23.334} + 30 + \frac{(-20 \times 20)}{23.334} = 8.571 \text{ mm}$$

$$\frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} = \frac{-200}{23.334} + 0 + \frac{20 \times 20}{23.334} = 8.571 \text{ mm}$$

$$[B] = \frac{1}{2 \times 300} \begin{bmatrix} -30 & 0 & 30 & 0 & 0 & 0 \\ 8.571 & 0 & 8.571 & 0 & 8.571 & 0 \\ 0 & 0 & 0 & -20 & 0 & 20 \\ 0 & -30 & -20 & 30 & 20 & 0 \end{bmatrix}$$

$$[B] = \begin{bmatrix} -0.05 & 0 & 0.05 & 0 & 0 & 0 \\ 0.0142 & 0 & 0.0142 & 0 & 0.0142 & 0 \\ 0 & 0 & 0 & -0.0333 & 0 & 0.0333 \\ 0 & -0.05 & -0.0333 & 0.05 & 0.0333 & 0 \end{bmatrix}$$

(2) For the element shown in figure determine the stiffness matrix. Take E = 200 GPa and $\nu = 0.25$



The co-ordinates shown in figure are in millimeters.

GIVEN DATA:

$r_1 = 0$ mm $\quad z_1 = 0$ mm

$r_2 = 50$ mm $\quad z_2 = 0$ mm

$r_3 = 50$ mm $\quad z_3 = 50$ mm

$E = 200$ GPa $= 200 \times 10^9$ N/m² $= 200 \times 10^3$ N/mm²

$E = 2 \times 10^5$ N/mm²

$\nu = 0.25$

TOFIND:

$[k]$

SOLUTION:

$$[k] = 2\pi r A [B]^T [D] [B]$$

$$A = \frac{1}{2}\begin{vmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{vmatrix}$$

(OR)

$$A = \frac{1}{2} b h = \frac{1}{2} \times 50 \times 50 = 1250 \ mm^2$$

$$r = \frac{r_1 + r_2 + r_3}{3} = \frac{0 + 50 + 50}{3} = 33.333 \ mm$$

$$z = \frac{z_1 + z_2 + z_3}{3} = \frac{0 + 0 + 50}{3} = 16.666 \ mm$$

$$[D] = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

$$= \frac{2 \times 10^5}{(1+0.25)(1-2\times 0.25)}\begin{bmatrix} 1-0.25 & 0.25 & 0.25 & 0 \\ 0.25 & 1-0.25 & 0.25 & 0 \\ 0.25 & 0.25 & 1-0.25 & 0 \\ 0 & 0 & 0 & \frac{1-2\times 0.25}{2} \end{bmatrix}$$

$$= 320\times10^3 \begin{bmatrix} 0.75 & 0.25 & 0.25 & 0 \\ 0.25 & 0.75 & 0.25 & 0 \\ 0.25 & 0.25 & 0.75 & 0 \\ 0 & 0 & 0 & 0.25 \end{bmatrix}$$

$$= 320\times10^3 \times 0.25 \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[D] = 80\times10^3 \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[B] = \frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 Z}{r} & 0 & \frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 Z}{r} & 0 & \frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 Z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$$\alpha_1 = r_2 Z_3 - r_3 Z_2 = (50\times50) - (50\times0) = 2500 \text{ mm}^2$$

$$\alpha_2 = r_3 Z_1 - r_1 Z_3 = (50\times0) - (0\times50) = 0$$

$$\alpha_3 = r_1 Z_2 - r_2 Z_1 = (0\times0) - (50\times0) = 0$$

$$\beta_1 = z_2 - z_3 = 0 - 50 = -50 \text{ mm}$$

$$\beta_2 = z_3 - z_1 = 50 - 0 = 50 \text{ mm}$$

$$\beta_3 = z_1 - z_2 = 0 - 0 = 0$$

$$\gamma_1 = r_3 - r_2 = 50 - 50 = 0$$

$$\gamma_2 = r_1 - r_3 = 0 - 50 = -50$$

$$\gamma_3 = r_2 - r_1 = 50 - 0 = 50 \text{ mm}$$

$$\frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} = \frac{2500}{33.333} + (-50) + 0 = 25 \text{ mm}$$

$$\frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} = 0 + 50 + \frac{(-50 \times 16.666)}{33.333} = 25 \text{ mm}$$

$$\frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} = 0 + 0 + \frac{(50 \times 16.666)}{33.333} = 25 \text{ mm}$$

$$[B] = \frac{1}{2 \times 1250} \begin{bmatrix} -50 & 0 & 50 & 0 & 0 & 0 \\ 25 & 0 & 25 & 0 & 25 & 0 \\ 0 & 0 & 0 & -50 & 0 & 50 \\ 0 & -50 & -50 & 50 & 50 & 0 \end{bmatrix}$$

$$[D][B] = 80 \times 10^3 \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \frac{1}{2 \times 1250} \begin{bmatrix} -50 & 0 & 50 & 0 & 0 & 0 \\ 25 & 0 & 25 & 0 & 25 & 0 \\ 0 & 0 & 0 & -50 & 0 & 50 \\ 0 & -50 & -50 & 50 & 50 & 0 \end{bmatrix}$$

$$= 800 \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 2 \\ 0 & -2 & -2 & 2 & 2 & 0 \end{bmatrix}$$

$$[D][B] = 800 \begin{bmatrix} -5 & 0 & 7 & -2 & 1 & 2 \\ 1 & 0 & 5 & -2 & 3 & 2 \\ -1 & 0 & 3 & -6 & 1 & 6 \\ 0 & -2 & -2 & 2 & 2 & 0 \end{bmatrix}$$

$$[B]^T = 0.01 \begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 2 & 1 & 0 & -2 \\ 0 & 0 & -2 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

$$[B]^T [D][B] = 0.01 \begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 2 & 1 & 0 & -2 \\ 0 & 0 & -2 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix} \times 800 \begin{bmatrix} -5 & 0 & 7 & -2 & 1 & 2 \\ 1 & 0 & 5 & -2 & 3 & 2 \\ -1 & 0 & 3 & -6 & 1 & 6 \\ 0 & -2 & -2 & 2 & 2 & 0 \end{bmatrix}$$

$$[B]^T[D][B] = 8\begin{bmatrix} 11 & 0 & -9 & 2 & 1 & -2 \\ 0 & 4 & 4 & -4 & -4 & 0 \\ -9 & 4 & 23 & -10 & 1 & 6 \\ 2 & -4 & -10 & 16 & 2 & -12 \\ 1 & -4 & 1 & 2 & 7 & 2 \\ -2 & 0 & 6 & -12 & 2 & 12 \end{bmatrix}$$

$$[K] = 2\pi \times 33.333 \times 1250 \times 8\begin{bmatrix} 11 & 0 & -9 & 2 & 1 & -2 \\ 0 & 4 & 4 & -4 & -4 & 0 \\ -9 & 4 & 23 & -10 & 1 & 6 \\ 2 & -4 & -10 & 16 & 2 & -12 \\ 1 & -4 & 1 & 2 & 7 & 2 \\ -2 & 0 & 6 & -12 & 2 & 12 \end{bmatrix}$$

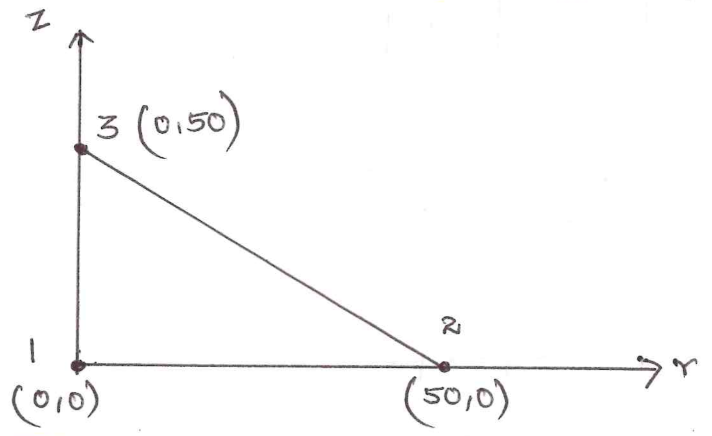
$$[K] = 2.094 \times 10^6\begin{bmatrix} 11 & 0 & -9 & 2 & 1 & -2 \\ 0 & 4 & 4 & -4 & -4 & 0 \\ -9 & 4 & 23 & -10 & 1 & 6 \\ 2 & -4 & -10 & 16 & 2 & -12 \\ 1 & -4 & 1 & 2 & 7 & 2 \\ -2 & 0 & 6 & -12 & 2 & 12 \end{bmatrix} N/mm$$

(3) For the axisymmetric elements shown in figure determine the element stresses. Let E = 210 GPa and $\nu = 0.25$. The co-ordinates (in mm) are shown in figure. The nodal displacements are

$u_1 = 0.05$ mm $\quad w_1 = 0.03$ mm

$u_2 = 0.02$ mm $\quad w_2 = 0.02$ mm

$u_3 = 0$ mm $\quad w_3 = 0$ mm



GIVEN DATA:

$r_1 = 0$ mm $\quad z_1 = 0$ mm

$r_2 = 50$ mm $\quad z_2 = 0$ mm

$r_3 = 0$ mm $\quad z_3 = 50$ mm

$E = 210$ GPa $= 2.1 \times 10^5$ N/mm^2

$\nu = 0.25$

$u_1 = 0.05$ mm $\quad w_1 = 0.03$ mm

$u_2 = 0.02$ mm $\quad w_2 = 0.02$ mm

$u_3 = 0$ mm $\quad w_3 = 0$ mm

TOFIND:

1. Radial stress σ_r
2. Circumferential stress σ_θ
3. Longitudinal stress σ_z
4. Shear stress τ_{rz}

SOLUTION:

$$[K] = 2\pi r A [B]^T [D] [B]$$

$$A = \frac{1}{2} b h = \frac{1}{2} \times 50 \times 50 = 1250 \text{ mm}^2$$

$$r = \frac{r_1 + r_2 + r_3}{3} = \frac{0+50+0}{3} = 16.667 \text{ mm}$$

$$z = \frac{z_1 + z_2 + z_3}{3} = \frac{0+0+50}{3} = 16.667 \text{ mm}$$

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

$$[D] = \frac{2.1 \times 10^5}{(1+0.25)(1-(2\times 0.25))} \begin{bmatrix} 0.75 & 0.25 & 0.25 & 0 \\ 0.25 & 0.75 & 0.25 & 0 \\ 0.25 & 0.25 & 0.75 & 0 \\ 0 & 0 & 0 & 0.25 \end{bmatrix}$$

$$= 336 \times 10^3 \times 0.25 \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[D] = 84 \times 10^3 \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[B] = \frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} & 0 & \frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} & 0 & \frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$$\alpha_1 = r_2 z_3 - r_3 z_2 = (50\times 50) - (0\times 0) = 2500 \text{ mm}^2$$

$$\alpha_2 = r_3 z_1 - r_1 z_3 = (0\times 0) - (0\times 0) = 0$$

$$\alpha_3 = r_1 z_2 - r_2 z_1 = (0\times 0) - (50\times 0) = 0$$

$$\beta_1 = z_2 - z_3 = 0 - 50 = -50 \text{ mm}$$

$$\beta_2 = z_3 - z_1 = 50 - 0 = 50 \text{ mm}$$

$$\beta_3 = z_1 - z_2 = 0 - 0 = 0$$

$$\gamma_1 = r_3 - r_2 = 0 - 50 = -50 \text{ mm}$$

$$\gamma_2 = r_1 - r_3 = 0 - 0 = 0$$

$$\gamma_3 = r_2 - r_1 = 50 - 0 = 50 \text{ mm}$$

$$\frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} = \frac{2500}{16.667} + (-50) + \frac{-50 \times 16.667}{16.667} = 50 \text{ mm}$$

$$\frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} = \frac{0}{16.667} + 50 + \frac{0 \times 16.667}{16.667} = 50 \text{ mm}$$

$$\frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} = \frac{0}{16.667} + 0 + \frac{50 \times 16.667}{16.667} = 50 \text{ mm}$$

$$[B] = \frac{1}{2 \times 1250} \begin{bmatrix} -50 & 0 & 50 & 0 & 0 & 0 \\ 50 & 0 & 50 & 0 & 50 & 0 \\ 0 & -50 & 0 & 0 & 0 & 50 \\ -50 & -50 & 0 & 50 & 50 & 0 \end{bmatrix}$$

$$[B] = 0.02 \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$[D][B] = 84\times10^{3}\begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times 0.02\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$[D][B] = 1680\begin{bmatrix} -2 & -1 & 4 & 0 & 1 & 1 \\ 2 & -1 & 4 & 0 & 3 & 1 \\ 0 & -3 & 2 & 0 & 1 & 3 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\{\sigma\} = [D][B]\{u\}$$

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{Bmatrix} = [D][B]\begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{Bmatrix}$$

$$= 1680\begin{bmatrix} -2 & -1 & 4 & 0 & 1 & 1 \\ 2 & -1 & 4 & 0 & 3 & 1 \\ 0 & -3 & 2 & 0 & 1 & 3 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}\begin{Bmatrix} 0.05 \\ 0.03 \\ 0.02 \\ 0.02 \\ 0 \\ 0 \end{Bmatrix}$$

$$= 1680 \begin{Bmatrix} -0.05 \\ 0.15 \\ -0.05 \\ -0.06 \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{Bmatrix} = \begin{Bmatrix} -84 \\ 252 \\ -84 \\ -100.8 \end{Bmatrix}$$

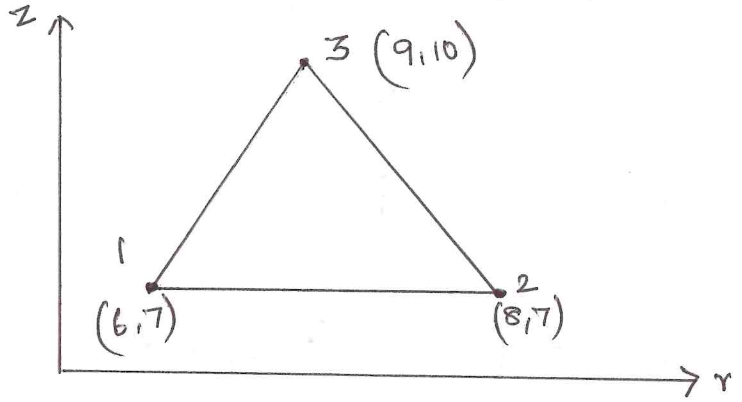
$\sigma_r = -84 \ N/mm^2$

$\sigma_\theta = 252 \ N/mm^2$

$\sigma_z = -84 \ N/mm^2$

$\tau_{rz} = -100.8 \ N/mm^2$

4. Calculate the element stiffness matrix and the thermal force vector for the axisymmetric triangular element shown in figure. The element experiences a 15°C increase in temperature. The co-ordinates are in millimeters. Take $\alpha = 10 \times 10^{-6} /°C$, $E = 2 \times 10^5 \ N/mm^2$, $\gamma = 0.25$



GIVEN DATA:

$\Delta T = 15°C$

$r_1 = 6 \ mm \qquad z_1 = 7 \ mm$

$r_2 = 8 \ mm \qquad z_2 = 7 \ mm$

$r_3 = 9 \ mm \qquad z_3 = 10 \ mm$

$\alpha = 10 \times 10^{-6} /°C$

$E = 2 \times 10^5 \ N/mm^2$

$\gamma = 0.25$

TO FIND:

$[k], \{F\}_t$

SOLUTION:

$$[k] = 2\pi r A [B]^T [D] [B]$$

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1}{r}+\beta_1+\frac{\gamma_1 z}{r} & 0 & \frac{\alpha_2}{r}+\beta_2+\frac{\gamma_2 z}{r} & 0 & \frac{\alpha_3}{r}+\beta_3+\frac{\gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$$\alpha_1 = r_2 z_3 - r_3 z_2 = (8\times 10) - (9\times 7) = 17\ mm^2$$

$$\alpha_2 = r_3 z_1 - r_1 z_3 = (9\times 7) - (6\times 10) = 3\ mm^2$$

$$\alpha_3 = r_1 z_2 - r_2 z_1 = (6\times 7) - (8\times 7) = -14\ mm^2$$

$$\beta_1 = z_2 - z_3 = 7-10 = -3\ mm$$

$$\beta_2 = z_3 - z_1 = 10-7 = 3\ mm$$

$$\beta_3 = z_1 - z_2 = 7-7 = 0$$

$$\gamma_1 = r_3 - r_2 = 9-8 = 1\ mm$$

$$\gamma_2 = r_1 - r_3 = 6-9 = -3\ mm$$

$$\gamma_3 = r_2 - r_1 = 8-6 = 2\ mm$$

$$r = \frac{r_1+r_2+r_3}{3} = \frac{6+8+9}{3} = 7.6667\ mm$$

$$z = \frac{z_1+z_2+z_3}{3} = \frac{7+7+10}{3} = 8\ mm$$

$$\frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} = \frac{17}{7.6667} + (-3) + \frac{1 \times 8}{7.6667} = 0.2609 \text{ mm}$$

$$\frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} = \frac{3}{7.6667} + 3 + \frac{-3 \times 8}{7.6667} = 0.2609 \text{ mm}$$

$$\frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} = \frac{-14}{7.6667} + 0 + \frac{2 \times 8}{7.6667} = 0.2609 \text{ mm}$$

$$A = \frac{1}{2} \begin{vmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{vmatrix}$$

$$= \frac{1}{2} \begin{vmatrix} 1 & 6 & 7 \\ 1 & 8 & 7 \\ 1 & 9 & 10 \end{vmatrix}$$

$$A = \frac{1}{2} \left[1 \, (8 \times 10) - (9 \times 7) - 6 \, (10 - 7) + 7 \, (9 - 8) \right]$$

$$A = 3 \text{ mm}$$

$$[B] = \frac{1}{2 \times 3} \begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0.2609 & 0 & 0.2609 & 0 & 0.2609 & 0 \\ 0 & 1 & 0 & -3 & 0 & 2 \\ 1 & -3 & -3 & 3 & 2 & 0 \end{bmatrix}$$

$$[B]^T = 0.1667 \begin{bmatrix} -3 & 0.2609 & 0 & 1 \\ 0 & 0 & 1 & -3 \\ 3 & 0.2609 & 0 & -3 \\ 0 & 0 & -3 & 3 \\ 0 & 0.2609 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

$$= \frac{2\times10^5}{(1+0.25)[1-(2\times0.25)]} \begin{bmatrix} 1-0.25 & 0.25 & 0.25 & 0 \\ 0.25 & 1-0.25 & 0.25 & 0 \\ 0.25 & 0.25 & 1-0.25 & 0 \\ 0 & 0 & 0 & \frac{1-2\times0.25}{2} \end{bmatrix}$$

$$[D] = 0.25 \times 320 \times 10^3 \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[B]^T[D] = 0.1667 \begin{bmatrix} -3 & 0.2609 & 0 & 1 \\ 0 & 0 & 1 & -3 \\ 3 & 0.2609 & 0 & -3 \\ 0 & 0 & -3 & 3 \\ 0 & 0.2609 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix} \times 0.25 \times 320 \times 10^3 \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[B]^T[D] = 13.336 \times 10^3 \begin{bmatrix} -8.7391 & -2.2173 & -2.7391 & 1 \\ 1 & 1 & 3 & -3 \\ 9.2609 & 3.7827 & 3.2609 & -3 \\ -3 & -3 & -9 & 3 \\ 0.2609 & 0.7827 & 0.2609 & 2 \\ 2 & 2 & 6 & 0 \end{bmatrix}$$

$$[B]^T[D][B] = 2.2231 \times 10^3 \begin{bmatrix} 26.6388 & -5.7391 & -29.7958 & 11.2173 & 1.4215 & -5.4782 \\ -5.7391 & 12 & 12.2609 & -18 & -5.7391 & 6 \\ -29.7958 & 12.2609 & 37.7696 & -18.7827 & -5.0131 & 6.5218 \\ 11.2173 & -18 & -18.7827 & 36 & 5.2173 & -18 \\ 1.4215 & -5.7391 & -5.0131 & 5.2173 & 4.2042 & 0.5218 \\ -5.4782 & 6 & 6.5218 & -18 & 0.5218 & 12 \end{bmatrix}$$

$$[K] = 321.2688 \times 10^3 \times \begin{bmatrix} 26.6388 & -5.7391 & -29.7958 & 11.2173 & 1.4215 & -5.4782 \\ -5.7391 & 12 & 12.2609 & -18 & -5.7391 & 6 \\ -29.7958 & 12.2609 & 37.7696 & -18.7827 & -5.0131 & 6.5218 \\ 11.2173 & -18 & -18.7827 & 36 & 5.2173 & -18 \\ 1.4215 & -5.7391 & -5.0131 & 5.2173 & 4.2042 & 0.5218 \\ -5.4782 & 6 & 6.5218 & -18 & 0.5218 & 12 \end{bmatrix}$$

$$\{F\}_t = [B]^T [D] \{e\}_t \times 2\pi r A$$

$$\text{strain } \{e\}_t = \begin{Bmatrix} \alpha \Delta T \\ \alpha \Delta T \\ 0 \\ \alpha \Delta T \end{Bmatrix} = \begin{Bmatrix} 10 \times 10^{-6} \times 15 \\ 10 \times 10^{-6} \times 15 \\ 0 \\ 10 \times 10^{-6} \times 15 \end{Bmatrix}$$

$$\{e\}_t = 10^{-6} \begin{Bmatrix} 150 \\ 150 \\ 0 \\ 150 \end{Bmatrix}$$

$$\{F\}_t = 13.336 \times 10^3 \begin{bmatrix} -8.7391 & -2.2173 & -2.7391 & 1 \\ 1 & 1 & 3 & -3 \\ 9.2609 & 3.7872 & 3.2609 & -3 \\ -3 & -3 & -9 & 3 \\ 0.2609 & 0.7827 & 0.2609 & 2 \\ 2 & 2 & 6 & 0 \end{bmatrix} \times$$

$$10^{-6} \times \begin{Bmatrix} 150 \\ 150 \\ 0 \\ 150 \end{Bmatrix} \times 2 \times \pi \times 7.6667 \times 3$$

$$= 1.927 \begin{Bmatrix} -1493.46 \\ -150 \\ 1506.54 \\ -450 \\ 456.54 \\ 600 \end{Bmatrix}$$

$$\{F\}_t = \begin{Bmatrix} -2878.25 \\ -289.08 \\ 2903.45 \\ -867.25 \\ 879.86 \\ 1156.34 \end{Bmatrix} N$$

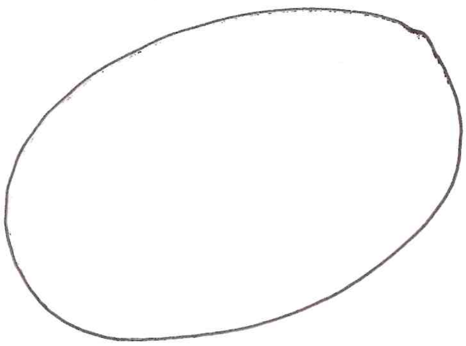
UNIT V

ISOPARAMETRIC ELEMENTS:

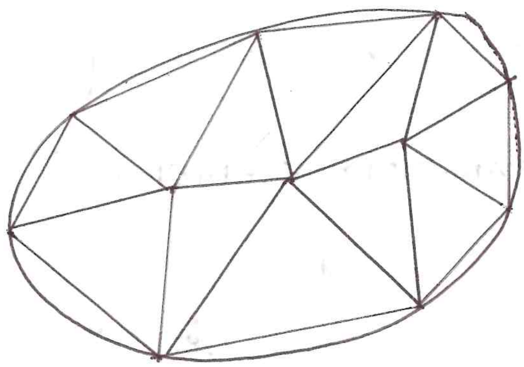
INTRODUCTION:

The finite element method is a powerful technique for analysing engineering problems involving complex and irregular geometrics. However the two and three dimensional elements (triangle, rectangle, brick) discussed in previous chapters cannot be used efficiently for irregular geometrics.

Consider a continuum shown in figure and it is discretized by using triangular elements.



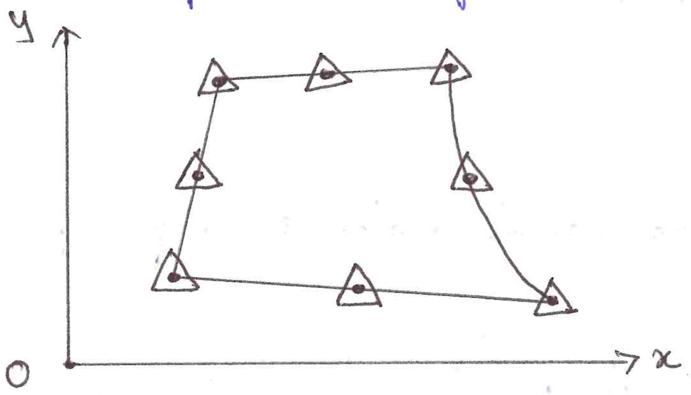
CONTINUUM



CONTINUUM IS DISCRETIZED BY TRIANGULAR ELEMENTS

ISOPARAMETRIC ELEMENT:

We know that shape functions are used for defining the geometry and displacements of the element.



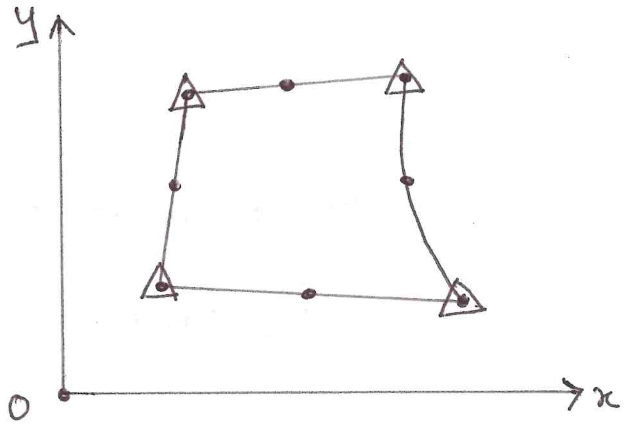
● Nodes used for defining geometry

△ Nodes used for defining displacements.

In this element all the eight nodes are used in defining geometry as well as displacements.

If the number of nodes used for defining the geometry is same as number of nodes used for defining the displacements then it is known as isoparametric element.

SUPERPARAMETRIC ELEMENT:

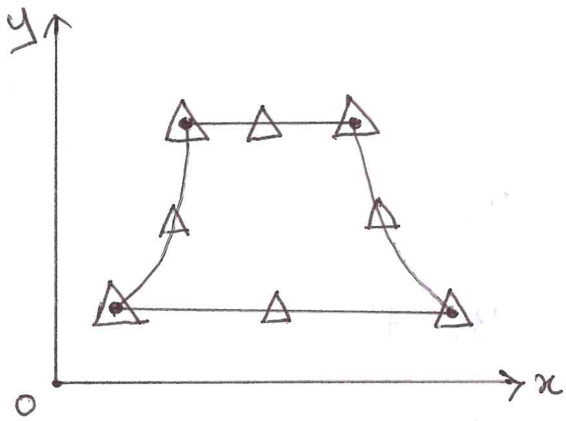


- ● Nodes used for defining geometry
- △ Nodes used for defining displacements.

In this element eight nodes are used to define the geometry and four nodes are used to define the displacements.

If the number of nodes used for defining the geometry is more than number of nodes used for defining the displacements then it is known as superparametric element.

SUBPARAMETRIC ELEMENT:

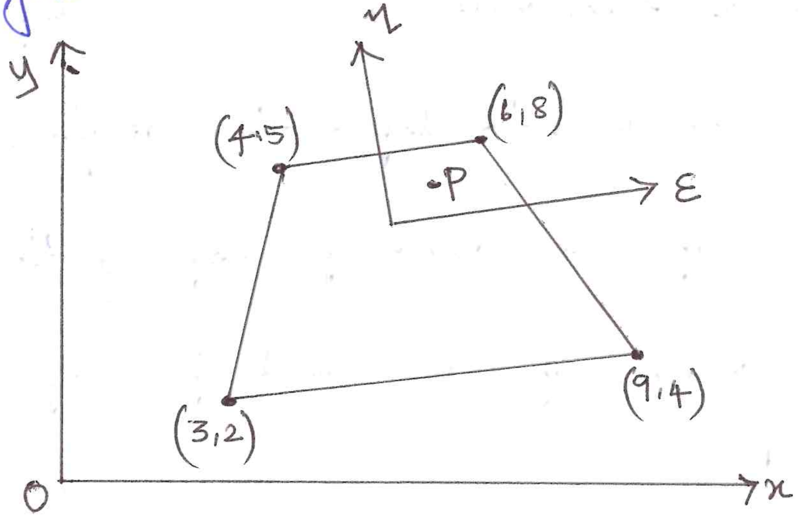


- ● Nodes used for defining geometry
- △ Nodes used for defining displacements

In this element four nodes are used to define the geometry and eight nodes are used to define the displacements.

If the number of nodes used for defining the geometry is less than number of nodes used for defining the displacements then it is known as subparametric element.

1) Evaluate the cartesian co-ordinate of the point P which has local co-ordinates $\varepsilon = 0.6$ and $\eta = 0.8$ as shown in figure.



GIVEN DATA:

$\varepsilon = 0.6$

$\eta = 0.8$

$x_1 = 3$; $y_1 = 2$

$x_2 = 9$; $y_2 = 4$

$x_3 = 6$; $y_3 = 8$

$x_4 = 4$; $y_4 = 5$

TOFIND:

Point $P(x, y)$

SOLUTION:

$$N_1 = \frac{1}{4}(1-\varepsilon)(1-\eta)$$

$$N_2 = \frac{1}{4}(1+\varepsilon)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\varepsilon)(1+\eta)$$

$$N_4 = \frac{1}{4}(1-\varepsilon)(1+\eta)$$

$$N_1 = \frac{1}{4}(1-0.6)(1-0.8) = 0.02$$

$$N_2 = \frac{1}{4}(1+0.6)(1-0.8) = 0.08$$

$$N_3 = \frac{1}{4}(1+0.6)(1+0.8) = 0.72$$

$$N_4 = \frac{1}{4}(1-0.6)(1-0.8) = 0.18$$

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$

$$= 0.02 \times 3 + 0.08 \times 9 + 0.72 \times 6 + 0.18 \times 4$$

$$x = 5.82$$

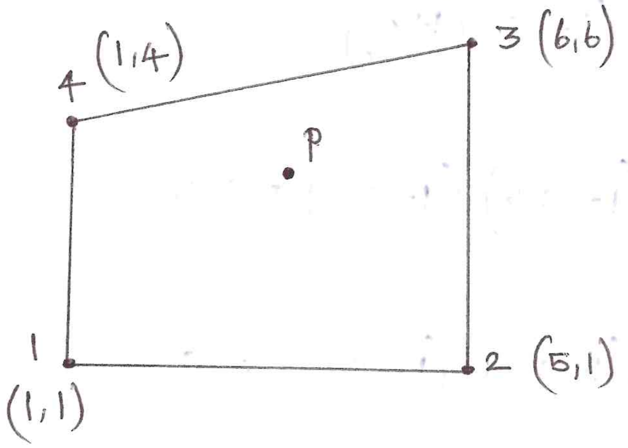
$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$$

$$= 0.02 \times 2 + 0.08 \times 4 + 0.72 \times 8 + 0.18 \times 5$$

$$y = 7.02$$

$$P = (\overset{x}{5.82}, \overset{y}{7.02})$$

② For the isoparametric four noded quadrilateral element shown in figure, determine the cartesian co-ordinates of point P which has local co-ordinates $\varepsilon = 0.5$ and $\eta = 0.5$



GIVEN DATA:

$x_1 = 1$; $y_1 = 1$

$x_2 = 5$ $y_2 = 1$

$x_3 = 6$ $y_3 = 6$

$x_4 = 1$ $y_4 = 4$

TOFIND:

P (x, y)

SOLUTION:

$$N_1 = \frac{1}{4}(1-\varepsilon)(1-\eta)$$

$$N_2 = \frac{1}{4}(1+\varepsilon)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\varepsilon)(1+\eta)$$

$$N_4 = \frac{1}{4}(1-\varepsilon)(1+\eta)$$

$$N_1 = \frac{1}{4}(1-0.5)(1-0.5) = 0.0625$$

$$N_2 = \frac{1}{4}(1+0.5)(1-0.5) = 0.1875$$

$$N_3 = \frac{1}{4}(1+0.5)(1+0.5) = 0.5625$$

$$N_4 = \frac{1}{4}(1-0.5)(1+0.5) = 0.1875$$

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$

$$= 0.0625 \times 1 + 0.1875 \times 5 + 0.5625 \times 6 + 0.1875 \times 1$$

$$x = 4.5625$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$$

$$= 0.0625 \times 1 + 0.1875 \times 1 + 0.5625 \times 6 + 0.1875 \times 4$$

$$y = 4.375$$

$$P = (\overset{x}{4.5625}, \overset{y}{4.375})$$

---

(3) For the isoparametric quadrilateral element shown in figure determine the local co-ordinates of the point P which has cartesian co-ordinates (7, 4)

GIVEN DATA:

Point P

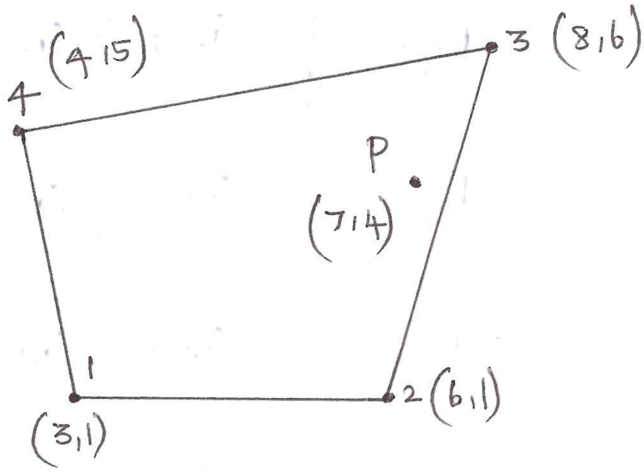
$x = 7 \, , \quad y = 4$

$x_1 = 3 \quad y_1 = 1$

$x_2 = 6 \quad y_2 = 1$

$x_3 = 8 \quad y_3 = 6$

$x_4 = 2 \quad y_4 = 5$



TOFIND:

ε , η

SOLUTION:

$$N_1 = \frac{1}{4}(1-\varepsilon)(1-\eta)$$

$$N_2 = \frac{1}{4}(1+\varepsilon)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\varepsilon)(1+\eta)$$

$$N_4 = \frac{1}{4}(1-\varepsilon)(1+\eta)$$

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4 \rightarrow (1)$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4 \rightarrow (2)$$

$$7 = \frac{1}{4}\Big[(1-\varepsilon)(1-\eta)\times 3 + (1+\varepsilon)(1-\eta)\times 6 + (1+\varepsilon)(1+\eta)\times 8 +$$

$$(1-\varepsilon)(1+\eta)\times 2\Big]$$

$$28 = \left[(1-\eta-\varepsilon+\varepsilon\eta)3 + (1-\eta+\varepsilon-\varepsilon\eta)6 + (1+\eta+\varepsilon+\varepsilon\eta)8 + (1+\eta-\varepsilon-\varepsilon\eta)2\right]$$

$$28 = 3 - 3\eta - 3\varepsilon + 3\varepsilon\eta + 6 - 6\eta + 6\varepsilon - 6\varepsilon\eta + 8 + 8\eta + 8\varepsilon + 8\varepsilon\eta + 2 + 2\eta - 2\varepsilon - 2\varepsilon\eta$$

$$28 = 19 + \eta + 9\varepsilon + 3\varepsilon\eta$$

$$\eta + 9\varepsilon + 3\varepsilon\eta = 9 \longrightarrow (3)$$

$$4 = \frac{1}{4}\left[(1-\varepsilon)(1-\eta)\times 1 + (1+\varepsilon)(1-\eta)\times 1 + (1+\varepsilon)(1+\eta)\times 6 + (1-\varepsilon)(1+\eta)\times 5\right]$$

$$= \frac{1}{4}\left[1-\eta-\varepsilon+\varepsilon\eta+1-\eta+\varepsilon-\varepsilon\eta+6+6\eta+6\varepsilon+6\varepsilon\eta+5+5\eta-5\varepsilon-5\varepsilon\eta\right]$$

$$4 = \frac{1}{4}\left[13+9\eta+\varepsilon+\varepsilon\eta\right]$$

$$16 = 13 + 9\eta + \varepsilon + \varepsilon\eta$$

$$9\eta + \varepsilon + \varepsilon\eta = 3 \longrightarrow (4)$$

Multiply eqn (4) by (-3)

$$-27\eta - 3\varepsilon - 3\varepsilon\eta = -9 \longrightarrow (5)$$

Solving equations (3) and (5)

$$\varepsilon = 4.3333\eta \rightarrow (6)$$

Substitute $\varepsilon = 4.3333\eta$ in eqn (3)

$$\eta + 9(4.3333\eta) + 3(4.3333\eta) \times \eta = 9$$

$$\eta + 39\eta + 13\eta^2 = 9$$

$$13\eta^2 + 40\eta = 9$$

$$13\eta^2 + 40\eta - 9 = 0$$

$$\eta = \frac{-40 \pm \sqrt{(40)^2 - 4(13)(-9)}}{2(13)}$$

$$= \frac{-40 + 45.475}{26}$$

$$\eta = 0.210587$$

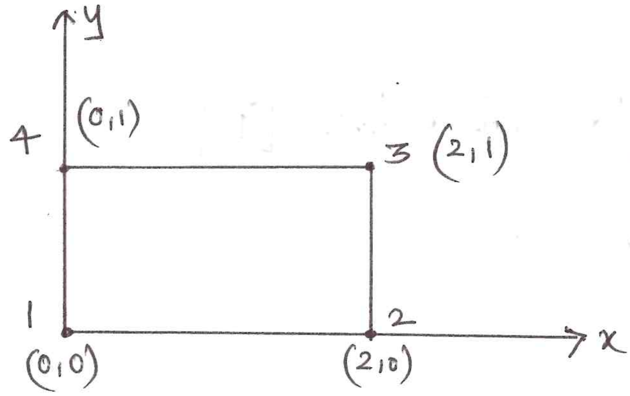
Substitute $\eta = 0.210587$ in eqn (6)

$$\varepsilon = 4.3333 \times 0.210587 = 0.912545$$

$$\eta = 0.210587$$

$$\varepsilon = 0.912545$$

4. A four noded rectangular element is shown in figure. Determine the following: 1. Jacobian matrix 2. Strain - Displacement matrix 3. Element stresses



Take $E = 2 \times 10^5 \ N/mm^2$, $\nu = 0.25$, $u = [0, 0, 0.003, 0.004, 0.006, 0.004, 0, 0]^T$

$\varepsilon = 0$, $\eta = 0$

Assume plane stress condition

<u>GIVEN DATA:</u>

$x_1 = 0 \qquad y_1 = 0$

$x_2 = 2 \qquad y_2 = 0$

$x_3 = 2 \qquad y_3 = 1$

$x_4 = 0 \qquad y_4 = 1$

$E = 2 \times 10^5 \ N/mm^2$

$\nu = 0.25$

$$u = \begin{Bmatrix} 0 \\ 0 \\ 0.003 \\ 0.004 \\ 0.006 \\ 0.004 \\ 0 \\ 0 \end{Bmatrix}$$

$\varepsilon = 0 \; ; \; \eta = 0$

TOFIND:

1. Jacobian matrix, J
2. Strain-Displacement matrix $[B]$
3. Element stress, σ

SOLUTION:

$$[J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

$$J_{11} = \frac{1}{4}\left[-(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4\right] \rightarrow (1)$$

$$J_{12} = \frac{1}{4}\left[-(1-\eta)y_1 + (1-\eta)y_2 + (1+\eta)y_3 - (1+\eta)y_4\right] \rightarrow (2)$$

$$J_{21} = \frac{1}{4}\left[-(1-\varepsilon)x_1 - (1+\varepsilon)x_2 + (1+\varepsilon)x_3 + (1-\varepsilon)x_4\right] \rightarrow (3)$$

$$J_{22} = \frac{1}{4}\left[-(1-\varepsilon)y_1 - (1+\varepsilon)y_2 + (1+\varepsilon)y_3 + (1-\varepsilon)y_4\right] \rightarrow (4)$$

$$J_{11} = \frac{1}{4}\left[0+2+2-0\right]$$

$$J_{11} = 1$$

$$J_{12} = \frac{1}{4}(0+0+1-1)$$

$$J_{12} = 0$$

$$J_{21} = \frac{1}{4}[0-2+2+0]$$

$$J_{21} = 0$$

$$J_{22} = \frac{1}{4}[-0-0+1+1]$$

$$J_{22} = 0.5$$

$$[J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

$$[J] = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$$

$$|J| = 1 \times 0.5 - 0 = 0.5$$

Strain - Displacement matrix for quadrilateral element is

$$[B] = \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix} \times$$

$$\frac{1}{4}\begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\varepsilon) & 0 & -(1+\varepsilon) & 0 & (1+\varepsilon) & 0 & (1-\varepsilon) & 0 \\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1-\varepsilon) & 0 & -(1+\varepsilon) & 0 & (1+\varepsilon) & 0 & (1-\varepsilon) \end{bmatrix}$$

$$[B] = \frac{1}{0.5}\begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0.5 & 0 \end{bmatrix} \times \frac{1}{4}\begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{0.5 \times 4}\begin{bmatrix} -0.5 & 0 & 0.5 & 0 & 0.5 & 0 & -0.5 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 \\ -1 & -0.5 & -1 & 0.5 & 1 & 0.5 & 1 & -0.5 \end{bmatrix}$$

$$[B] = 0.25 \begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & -2 & 0 & 2 & 0 & 2 \\ -2 & -1 & -2 & 1 & 2 & 1 & 2 & -1 \end{bmatrix}$$

$$\sigma = [D][B]\{u\}$$

$$[D] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

$$= \frac{2 \times 10^5}{1-0.25^2} \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \frac{1-0.25}{2} \end{bmatrix}$$

$$= 213.33 \times 10^3 \times 0.25 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

$$[D] = 53.333 \times 10^3 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

$$\{\sigma\} = 53.333 \times 10^3 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} \times$$

$$0.25 \begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & -2 & 0 & 2 & 0 & 2 \\ -2 & -1 & -2 & 1 & 2 & 1 & 2 & -1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0.003 \\ 0.004 \\ 0.006 \\ 0.004 \\ 0 \\ 0 \end{Bmatrix}$$

$$= 53.333 \times 10^3 \times 0.25 \begin{bmatrix} -4 & -2 & 4 & -2 & 4 & 2 & -4 & 2 \\ -1 & -8 & 1 & -8 & 1 & 8 & -1 & 8 \\ -3 & -1.5 & -3 & 1.5 & 3 & 1.5 & 3 & -1.5 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0.003 \\ 0.004 \\ 0.006 \\ 0.004 \\ 0 \\ 0 \end{Bmatrix}$$

$$= 53.333 \times 10^3 \begin{Bmatrix} 0.036 \\ 0.009 \\ 0.021 \end{Bmatrix}$$

$$\{\sigma\} = \begin{Bmatrix} 480 \\ 120 \\ 280 \end{Bmatrix} N/m^2$$

NUMBER OF POINTS, n	LOCATION, x_i	CORRESPONDING WEIGHTS, w_i
1	$x_1 = 0.000$	2.000
2	$x_1, x_2 = \pm\sqrt{\frac{1}{3}}$ $= \pm 0.577350269189$	1.000
3	$x_1, x_3 = \pm\sqrt{\frac{3}{5}}$ $= \pm 0.774596669241$ $x_2 = 0.000$	$\frac{5}{9} = 0.55555$ $\frac{8}{9} = 0.88888$
4	$x_1, x_4 = \pm 0.8611363116$ $x_2, x_3 = \pm 0.3399810436$	0.3478548451 0.6521451549

(5) Evaluate $\int_{-1}^{1} (x^4 + x^2)\, dx$ by applying 3 point Gaussian quadrature.

GIVEN DATA:

$$I = \int_{-1}^{1} (x^4 + x^2)\, dx$$

$$f(x) = x^4 + x^2$$

TOFIND:

Evaluate the integral by using Gaussian quadrature with three Gauss points.

SOLUTION:

$$x_1 = \sqrt{\frac{3}{5}} = 0.774596669$$

$$x_2 = 0$$

$$x_3 = -\sqrt{\frac{3}{5}} = -0.774596669$$

$$W_1 = \frac{5}{9} = 0.55555$$

$$W_2 = \frac{8}{9} = 0.888888$$

$$W_3 = \frac{5}{9} = 0.55555$$

$f(x) = x^4 + x^2$

$f(x_1) = (x_1)^4 + (x_1)^2$

$= (0.774596669)^4 + (0.774596669)^2$

$f(x_1) = 0.96$

$w_1 f(x_1) = 0.55555 \times 0.96 = 0.5333$

$f(x_2) = (x_2)^4 + (x_2)^2$

$= 0^4 + 0^2$

$f(x_2) = 0$

$w_2 f(x_2) = 0.888888 \times 0 = 0$

$f(x_3) = (x_3)^4 + (x_3)^2$

$= (-0.774596669)^4 + (-0.774596669)^2$

$f(x_3) = 0.96$

$w_3 f(x_3) = 0.55555 \times 0.96 = 0.5333$

$w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3) = 0.5333 + 0 + 0.5333 = 1.0666$

$$\int_{-1}^{1} (x^4 + x^2)\, dx = 1.0666$$

VERIFICATION:

$$\int_{-1}^{1} (x^4 + x^2)\, dx = \left[\frac{x^5}{5}\right]_{-1}^{1} + \left[\frac{x^3}{3}\right]_{-1}^{1}$$

$$= \frac{1}{5} + \frac{1}{5} + \frac{1}{3} + \frac{1}{3}$$

$$= \frac{2}{5} + \frac{2}{3}$$

$$= \frac{6 + 10}{15}$$

$$= \frac{16}{15}$$

$$\int_{-1}^{1} (x^4 + x^2)\, dx = 1.0666$$

---

(6) Evaluate the integral by using Gaussian quadrature $\int_{-1}^{1} x^2\, dx$

GIVEN DATA:

$$I = \int_{-1}^{1} x^2\, dx$$

$$f(x) = x^2$$

TO FIND:

Evaluate the integral

SOLUTION:

$$2n-1 = 2$$

$$2n = 3$$

$$n = \frac{3}{2} = 1.5 \triangleq 2$$

$$x_1 = +\sqrt{\frac{1}{3}} = 0.577350269$$

$$x_2 = -\sqrt{\frac{1}{3}} = -0.577350269$$

$$w_1 = 1$$

$$w_2 = 1$$

$$f(x_1) = x_1^2 = (0.577350269)^2$$

$$f(x_1) = 0.333333333$$

$$w_1 f(x_1) = 1 \times 0.333333333 = 0.333333333$$

$$f(x_2) = x_2^2 = (-0.577350269)^2$$

$$f(x_2) = 0.3333333333$$

$$w_2 f(x_2) = 1 \times 0.3333333333 = 0.3333333333$$

$$w_1 f(x_1) + w_2 f(x_2) = 0.333333333 + 0.333333333$$

$$\int_{-1}^{1} x^2\,dx = 0.666666666$$

7) Evaluate the integral $I = \int_{-1}^{1}\int_{-1}^{1} (2x^2+3xy+4y^2)\, dx\, dy$ using Gauss integration.

GIVEN DATA:

$$I = \int_{-1}^{1}\int_{-1}^{1} (2x^2+3xy+4y^2)\, dx\, dy$$

$$f(x,y) = 2x^2+3xy+4y^2$$

TOFIND:

Evaluate the integral

SOLUTION:

$$2n-1 = 2$$

$$n = 1.5 \simeq 2$$

$$n = 2$$

$$x_1 = 0.57735$$

$$x_2 = -0.57735$$

$$y_1 = 0.57735$$

$$y_2 = -0.57735$$

$$W_1 = 1$$

$$W_2 = 1$$

$$\int_{-1}^{1}\int_{-1}^{1} f(x,y)\, dx\, dy = W_1^2 f(x_1,y_1) + W_1 W_2 f(x_1,y_2) + W_2 W_1 f(x_2,y_1) + W_2^2 f(x_2,y_2)$$

$$f(x,y) = 2x^2+3xy+4y^2$$

$$w_1^2 f(x_1,y_1) = w_1^2\left(2x_1^2+3x_1y_1+4y_1^2\right)$$

$$= 1^2\left[2(0.57735)^2+3\times 0.57735\times +0.57735 + 4(-0.57735)^2\right]$$

$$w_1^2 f(x_1,y_1) = 3$$

$$w_1w_2 f(x_1,y_2) = w_1w_2\left(2x_1^2+3x_1y_2+4y_2^2\right)$$

$$= 1\times 1\left[2\times(0.57735)^2+3\times 0.57735\times -0.57735+4(-0.57735)^2\right]$$

$$w_1w_2 f(x_1,y_2) = 1$$

$$w_2w_1 f(x_2,y_1) = w_2w_1\left(2x_2^2+3x_2y_1+4y_1^2\right)$$

$$= 1\times 1\left[2(-0.57735)^2+3\times -0.57735\times 0.57735+4(0.57735)^2\right]$$

$$w_2w_1 f(x_2,y_1) = 1$$

$$w_2^2 f(x_2,y_2) = w_2^2\left(2x_2^2+3x_2y_2+4y_2^2\right)$$

$$= 1^2\left[2(-0.57735)^2+3(-0.57735)(0.57735)+4(-0.57735)^2\right]$$

$$w_2^2 f(x_2,y_2) = 3$$

$$\int_{-1}^{1}\int_{-1}^{1}\left(2x^2+3xy+4y^2\right)dx\,dy = 3+1+1+3 = 8$$

VERIFICATION:

$$\int_{-1}^{1}\int_{-1}^{1}(2x^2+3xy+4y^2)\,dx\,dy = \int_{-1}^{1}\left[\frac{2}{3}x^3+\frac{3}{2}x^2y+4y^2x\right]_{-1}^{1}dy$$

$$= \int_{-1}^{1}\left\{\frac{2}{3}(1+1)+\frac{3}{2}y(1-1)+4y^2(1+1)\right\}dy$$

$$= \int_{-1}^{1}\left[\frac{4}{3}+8y^2\right]dy$$

$$= \int_{-1}^{1}\frac{4}{3}\,dy + \int_{-1}^{1}8y^2\,dy$$

$$= \left[\frac{4}{3}y\right]_{-1}^{1} + \left[\frac{8y^3}{3}\right]_{-1}^{1} = \frac{4}{3}(1+1)+\frac{8}{3}(1+1)$$

$$= \frac{8}{3}+\frac{16}{3}$$

$$= \frac{24}{3}$$

$$= 8$$

$$\int_{-1}^{1}\int_{-1}^{1}(2x^2+3xy+4y^2)\,dx\,dy = 8$$